Department of Mathematics
University of Fribourg (Switzerland)

BROAD AND NARROW REGIONS OF HYPERBOLIC STRUCTURES ON SURFACES

THESIS

presented to the Faculty of Science of the University of Fribourg (Switzerland) in consideration for the award of the academic grade of *Doctor scientiarum mathematicarum*

by

Federica Fanoni

from

Sondrio, Italy

Thesis No: 1906
UniPrint
2015

Accepted by the Faculty of Science of the University of Fribourg (Switzerland) upon the recommendation of:

Prof. Hugo Parlier (Thesis supervisor)
University of Fribourg, Switzerland

Prof. Ara Basmajian (Examiner)
CUNY Graduate Center and Hunter College, United States of America

Prof. Gregory McShane (Examiner)
Institut Fourier, France

Prof. Stefan Wenger (President of the Jury)
University of Fribourg, Switzerland

Fribourg, June 18th, 2015

Thesis supervisor	Dean
Hugo Parlier	Fritz Müller

Acknowledgements

First of all, I would like to express all my gratitude to my advisor Hugo Parlier. Thank you for introducing me to such a beautiful subject and for teaching me so much. Thank you for your enthusiasm, for your mathematical assistance, but also – and so importantly – for your human support. I knew I could count on you, especially when being a PhD student was tough. Thank you very much.

I am grateful to Ara Basmajian and Greg McShane for agreeing to be part of my jury, for the effort they put in reading my thesis and for taking the time to travel and be present at my defence. My thanks also go to Stefan Wenger for accepting to be the president of my jury.

I would like to thank all the colleagues in Fribourg who made the department of Mathematics a nice and friendly environment in which to work. I am also grateful to the CUNY Graduate Center, and in particular to Ara Basmajian, for hosting me during the last months of my PhD.

These years would have been much less fun and for sure not as enjoyable without all the wonderful people I had the chance of meeting. I feel lucky knowing that I have so many friends around the world. So thanks (merci, *shukran*, danke, grazie, tack) to (in no particular order) Ammar and Jordane (Dude and Dudette [Mike]), Camille (Camilley) and Flora, Daniele, Matthieu (the Gendulpho), Scott, Nicolas (Nicky), Ivan and Ornella, Elia, Martin, Kevin and Sara, Ann, Viveka, Chris, Blanca and Max.

La lista dei motivi per ringraziare i miei genitori, Dario e Marilisa, e mio fratello, Luca, è troppo lunga per questa pagina. Dirò soltanto che il vostro affetto, il vostro supporto costante e il vostro gioire con me per tutte le belle novità hanno reso il mio dottorato molto più semplice e felice. Grazie per tutto questo e perché so che ci sarete sempre.

Bram, je maakt mijn hele leven beter. Dank je dat je er was in deze jaren: ik vraag me af hoe ik het zonder jou zou hebben gedaan.

Abstract

This dissertation is dedicated to the study of some concrete properties of hyperbolic structures on surfaces. More precisely, it is focused on three main problems.

The first one is the study of the maximum injectivity radius, the radius of a largest possible embedded metric disk. We prove that there exists a universal (explicit) constant ρ_T such that each hyperbolic two-dimensional orbifold has maximum injectivity radius at least ρ_T and we determine the unique orbifold with maximum injectivity radius equal to ρ_T. We use this result to deduce that each surface has a point which is displaced at distance at least $2\rho_T$ by any non-trivial automorphism and to give an alternative characterization of surfaces with a maximal amount of symmetries.

The other two problems are about systoles (shortest simple closed geodesics) of hyperbolic surfaces. We are interested in bounding the length and the number of systoles. The first result is an upper bound on the systole length for finite area surfaces which does not depend on the number of cusps. The second result is an upper bound on the kissing number (the number of systoles) which grows linearly in the number of cusps and subquadratically in the genus. To obtain this bound, we study intersection properties of systoles. In particular, we prove that two systoles can intersect at most twice and that if they do intersect twice there is a constraint on their topological configuration.

The tools used in this thesis are those of hyperbolic trigonometry and topological and geometric properties of simple closed curves and geodesics.

Sommario

Questa dissertazione tratta di alcune proprietà concrete di strutture iperboliche su superfici. Più precisamente, è concentrata su tre problemi fondamentali.

Il primo è lo studio del raggio di iniettività massimo, il raggio del più grande disco metrico immerso. Mostriamo che esiste una costante universale (esplicita) ρ_T tale che ogni orbifold iperbolico bidimensionale ha raggio di iniettività massimo maggiore o uguale a ρ_T e determiniamo l'unico orbifold con raggio di iniettività massimo uguale a ρ_T. Utilizziamo questo risultato per dedurre che ogni superficie possiede un punto che è trasportato a distanza almeno $2\rho_T$ da ogni automorfismo diverso dall'identità e per dare una caratterizzazione alternativa delle superfici con il massimo numero possibile di simmetrie.

Gli altri due problemi trattano delle sistole (le più corte geodetiche chiuse e semplici) di superfici iperboliche. Siamo interessati a maggiorare la lunghezza e il numero delle sistole. Il primo risultato è un limite superiore per la lunghezza della sistole per superfici di area finita che non dipende dal numero di cuspidi. Il secondo risultato è un limite superiore per il numero di sistole (kissing number) che cresce linearmente all'aumentare del numero di cuspidi ed è subquadratico al variare del genere. Per ottenere questa maggiorazione, studiamo le proprietà di intersezione delle sistole. In particolare, dimostriamo che due sistole possono avere al massimo due punti di intersezione e che se si intersecano in due punti c'è una restrizione sulla loro configurazione topologica.

Gli strumenti utilizzati in questa tesi sono la trigonometria iperbolica e le proprietà topologiche e geometriche di curve e geodetiche chiuse e semplici.

Contents

CHAPTER 1

Introduction

Hyperbolic, Euclidean and spherical geometry are the three model geometries which correspond to Riemannian metrics of constant curvature, either positive (spherical geometry), zero (Euclidean geometry) or negative (hyperbolic geometry). One of the main aspects that distinguishes them is the behavior of lines (geodesics): in the Euclidean plane, two lines with a common orthogonal will stay at the same distance from each other, while in the hyperbolic plane they will diverge and on the (round) sphere they will converge. In more ancient terms, Euclidean geometry satisfies the fifth postulate of Euclid, which can be rephrased as

Given a line γ and a point P outside of it,
there exists a unique line through P and parallel to γ.

This does not hold for spherical or hyperbolic geometry: there is no such parallel line in spherical geometry and there exist more than one (actually, infinitely many) in hyperbolic geometry.

In the study of surfaces, hyperbolic geometry plays a special role. Indeed, if we want to endow a surface of negative Euler characteristic with a Riemannian metric of constant curvature, it follows from Gauss–Bonnet Theorem that the curvature should be negative. As most surfaces (all but a handful) have negative Euler characteristic, hyperbolic geometry turns out to be very prominent in dimension two. Even more so because, given a topological surface of negative Euler characteristic, there are (uncountably) many hyperbolic metrics that can be put on it, even if we restrict to smooth, finite area ones[1]. Many years of research have been spent to reach a better understanding of these structures and their two natural parameter spaces, Teichmüller and moduli spaces, in particular through the study of simple closed curves on surfaces. One of the classical results in this line of thought is Fenchel and Nielsen's construction of global coordinates for Teichmüller space, based on simple closed curves (see [**FN03**]). More recently, Thurston used simple closed curves and related objects to describe deformations of hyperbolic surfaces and to define a boundary of Teichmüller space (see for

[1]Unless we were so unlucky to start with the unique counterexample to this statement: the thrice punctured sphere.

instance [**Thu88**], [**Thu98**]). Among the authors who studied simple closed geodesics on hyperbolic surfaces, the work of this dissertation is especially related to Buser's and Schmutz Schaller's work. We will discuss some of their results and more background in the next chapters.

This thesis is divided into two main parts.

The first part (chapters 3 and 4) is dedicated to the study of maximal size disks isometrically embedded in hyperbolic surfaces and hyperbolic two-dimensional orbifolds. In particular, we are interested in finding a sharp lower bound for the radius of a maximal embedded metric disk (the *maximum injectivity radius*) of hyperbolic two-dimensional orbifolds. We prove the following result.

THEOREM A. *There exists and explicit constant ρ_T ($\rho_T \approx 0.187728\dots$) such that the maximum injectivity radius of any hyperbolic two-dimensional orbifold is at least ρ_T, with equality if and only if the orbifold is the sphere with three cone points of order* 2, 3 *and* 7.

Using this theorem, we deduce a result about hyperbolic surfaces and their group of orientation preserving isometries (denoted $\mathrm{Aut}^+(S)$).

THEOREM B. *For any hyperbolic surface S, there exists a point $p \in S$ which is displaced at distance at least $2\rho_T$ by any $\varphi \in Aut^+(S) \setminus \{\mathrm{id}\}$.*
Moreover, S is a Hurwitz surface (a closed surface with maximum number of self-isometries) if and only if for every $\rho > \rho_T$ and for every $p \in S$ there exists $\varphi \in Aut^+(S) \setminus \{\mathrm{id}\}$ such that $d(p, \varphi(p)) < 2\rho$.

We also show how the techniques of the proof of Theorem A can be used to give a new short proof of Theorem 4.1, a sharp lower bound for the maximum injectivity radius of finite area hyperbolic surfaces, due to Yamada (see [**Yam82**]).

The second part (chapters 5 and 6) is dedicated to the study of systoles of hyperbolic surfaces and in particular to two basic problems: bounding the length of systoles and the *kissing number* (the number of systoles).

In chapter 5, we prove an upper bound on the systole length of finite area surfaces which does not depend on the number of cusps.

THEOREM C. *There exists a universal constant $K < 8$ such that any finite area hyperbolic surface S of genus $g \geq 1$ has systole length $\mathrm{sys}(S)$ satisfying*

$$\mathrm{sys}(S) \leq 2 \log g + K.$$

One of the main results of chapter 6 is an upper bound for kissing numbers of finite area hyperbolic surfaces, depending on the systole length.

THEOREM D. *Let S be a hyperbolic surface of signature (g,n), of systole length* $\mathrm{sys}(S)=\ell$*. Then its kissing number* $\mathrm{Kiss}(S)$ *satisfies*

$$\mathrm{Kiss}(S) \leq 20\cosh\frac{\ell}{4} + 200\frac{e^{\ell/2}}{\ell}(2g-2+n).$$

Combining this results with the upper bounds for the length of the systole of Theorem C and the one of Schmutz Schaller in [**Sch94**] (Theorem 5.4), we obtain the following upper bound, independent on the length of the systole.

THEOREM E. *There exists a universal constant $C>0$ such that any hyperbolic surface of signature (g,n) with $g \geq 1$ satisfies*

$$\mathrm{Kiss}(S) \leq C(g+n)\frac{g}{\log(g+1)}.$$

If S is a n-punctured sphere, then

$$\mathrm{Kiss}(S) \leq \frac{7}{2}n - 5.$$

To prove these results about the kissing number, we also study the intersection properties of systoles on hyperbolic surfaces. The results we obtain can be summarized as follows.

THEOREM F. *Let S be a hyperbolic surface of signature $(g,n) \neq (0,3)$. If α and β are systoles on S, then*

$$i(\alpha,\beta) \leq 2$$

and if $i(\alpha,\beta)=2$, then either α or β surrounds two cusps. Furthermore, for every genus $g \geq 0$ there exists $n(g) \in \mathbb{N}$ and a surface S of signature $(g,n(g))$ which has systoles that intersect twice.

NOTE: most of the results discussed in this dissertation can be found in [**Fan15**] and [**FP14**].

The work of this thesis has been supported by the Swiss National Science Foundation grant number PP00P2_128557 and by the U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “RNMS: Geometric Structures and Representation Varieties” (the GEAR Network).

CHAPTER 2

Preliminaries

In this chapter we introduce the basic definitions of hyperbolic structures on surfaces and we present some of their properties. We do not have any pretence of completeness; rather, we would like to mention the basic tools needed for the work to come and to give good references where these tools are fully explained.

The main object of our interest are hyperbolic surfaces. A *(hyperbolic) surface* is a connected surface with a complete, finite area *hyperbolic metric* (i.e. a Riemannian metric of constant curvature -1). Unless otherwise stated, we will assume surfaces to be orientable. Each hyperbolic surface is locally isometric to the hyperbolic plane $\mathbb{H}^2$; models for this space are described in section 2 of this chapter.

A more general class of surfaces with a hyperbolic structure is the one of cone-surfaces. A *cone-surface* is a two-dimensional connected manifold, possibly with boundary, that can be triangulated by finitely many hyperbolic triangles; if it has boundary, we require it to be geodesic. A *cone point* is a point where the surface is not smooth. The *cone angle* at a cone point p is the sum of all angles at p of triangles containing p. An *admissible cone-surface* is a cone-surface such that all cone points have cone angle at most π. If a cone point has cone angle $\frac{2\pi}{k}$, for some positive integer k, we say that it has *order k*. A *(2-dimensional hyperbolic) orbifold* is an admissible cone-surface without boundary such that each cone point has integer order.

The *signature* of a cone-surface is the triple (g, n, b), where g is the genus, n is the number of *singular points* (i.e. cusps and cone points) and b is the number of boundary components. If $b = 0$, we will simply write (g, n).

Given a topological surface Σ of signature (g, n) with Euler characteristic $\chi(\Sigma) = 2 - 2g - n < 0$, there are many possible hyperbolic structures on it. We denote by $\mathcal{M}_{g,n}$ the *moduli space* of Σ, i.e. the set of all complete, finite area hyperbolic metrics on Σ, up to isometry. The *orbifold moduli space* $\mathcal{O}_{g,n}$ is the space of all orbifolds of signature (g, n), up to isometry. In particular, since all hyperbolic surfaces are orbifolds, $\mathcal{M}_{g,n} \subseteq \mathcal{O}_{g,n}$.

NOTATION: we will use $d_M(\ ,\ )$ for the distance on a metric space M and $D_r(p)$ for the set of points at distance at most r from a point $p \in M$:

$$D_r(p) = \{q \in M \mid d_M(p,q) < r\}.$$

Given a curve γ on M, we denote its length by $\ell_M(\gamma)$. If the metric space that we are considering is clear from the context, we will simply use $d(\ ,\ )$ instead of $d_M(\ ,\ )$ and $\ell(\,)$ instead of $\ell_M(\,)$.

1. An interlude on orbifolds

We have defined (2-dimensional hyperbolic) orbifolds as two-dimensional connected manifolds without boundary which can be triangulated by finitely many hyperbolic triangles, with the property that each cone point has integer order *(definition 1)*. This is a very explicit definition, based on simple and well understood objects (hyperbolic triangles). Orbifolds though are studied in a much more general context, without constraint on the dimension or on the geometric structure - see for instance [**BH99**, Chapter III.$\mathcal{G}$]. Roughly speaking, orbifolds are defined to be topological manifolds that are locally the quotients of simply connected manifolds by the action of finite groups. If the simply connected manifold carries a geometric structure which is respected by the action of the finite groups, an orbifold can be endowed with a geometric structure as well. Following this point of view, a (2-dimensional hyperbolic) orbifold *(definition 2)* is a topological surface S with a *hyperbolic orbifold structure*, that is:

(i) an open cover $\{U_i\}_{i\in I}$ of S,
(ii) for each $i \in I$, an open connected subset X_i of the hyperbolic plane, a finite subgroup Γ_i of orientation preserving isometries of $\mathbb{H}^2$ fixing X_i and a continuous map $q_i : X_i \to U_i$ (a *chart*) such that q_i induces a homeomorphism between X_i/Γ_i and U_i,
(iii) for all $x_i \in X_i$ and $x_j \in X_j$ such that $q_i(x_i) = q_j(x_j)$, an isometry h of $\mathbb{H}^2$ (called *change of chart*) and a neighborhood W of x_i such that

$$q_j \circ h = q_i$$

on W.

If all groups Γ_i are reduced to the identity (or more generally if they act freely), we obtain a hyperbolic surface. Allowing these groups to have fixed points is exactly what creates cone points.

Even though definition 2 can be generalized to many other contexts, it is probably not the easiest to deal with. Another and more concrete way to think about 2-dimensional hyperbolic orbifolds is to see them as quotients of hyperbolic surfaces, as follows. Given a hyperbolic surface S, its *automorphism group* $\mathrm{Aut}^+(S)$ is the group of orientation preserving isometries of the surface. We can define a (2-dimensional hyperbolic) orbifold *(definition 3)* to be the quotient of a hyperbolic surface S by the action of a subgroup Γ of $\mathrm{Aut}^+(S)$. Even though less explicit than definition 1, this third definition shows how we are naturally led to study orbifolds if we are interested in hyperbolic surfaces and their symmetries. Moreover, hyperbolic surfaces can be obtained as quotients of the hyperbolic plane by the action of some discrete group of isometries of $\mathbb{H}^2$. Thus definition 3 easily implies that also orbifolds are quotient of $\mathbb{H}^2$ by the action of some discrete group of isometries. While for surfaces the group action should be free, this is not required for orbifolds, and cone points appear when we have fixed points.

These three definitions are equivalent, and we dedicate the rest of this section to proving it.

Definition 1 ⇒ Definition 2
Suppose S satisfies definition 1. We can choose a cover of S given by simply connected disks $D_i = \{x \in S \,|\, d(x, x_i) < r_i\}$, for some $x_i \in S$ and $r_i > 0$, such that

(i) each disks contains at most one cone point, and
(ii) if D_i contains a cone point y, then $x_i = y$.

For every i, X_i can be taken to be a disk of radius r_i in $\mathbb{H}^2$. If D_i doesn't contain a cone point, we can set $\Gamma_i = \{\mathrm{id}\}$ and q_i to be an isometry between X_i and D_i. If D_i contains a cone point of order k, we define Γ_i to be the group generated by a rotation of order k with center the center of the disk and q_i to be the projection map. Since all charts are branched covering maps and local isometries outside of the cone points, each change of charts can be chosen to be an isometry of the hyperbolic plane. This defines a hyperbolic orbifold structure on S, thus S satisfies definition 2.

Definition 2 ⇒ Definition 3
For this implication we refer to the following theorem in [**BH99**, Chapter III.$\mathcal{G}$]:

THEOREM 2.1. *Let Q be a connected orbifold with a (G, Y)-geometric structure. Then there is a subgroup Γ of G acting properly on a connected manifold M endowed with a (G, Y)- structure such that Q with its (G, Y)-structure is naturally the quotient of M by the action of Γ.*

In our case, the $G = \mathrm{Isom}^+(\mathbb{H}^2)$ and $Y = \mathbb{H}^2$, so that Q is a 2-dimensional hyperbolic orbifold as in definition 2 and M is a hyperbolic surface.

Definition 3 ⇒ Definition 1
Suppose that S is the quotient of a hyperbolic surface F by the action of a subgroup Γ of $\mathrm{Aut}^+(F)$. Since by Hurwitz theorem $\mathrm{Aut}^+(F)$ is finite, the possible non-smooth points of S are cone points of integer order. Moreover, consider a Γ-invariant triangulation T of F with vertex set given by the cusps of F and the fixed points of Γ. Then the quotient triangulation T/Γ gives a decomposition of S in triangles as requested. All isometries of Γ are orientation preserving and F doesn't have boundary, so S doesn't have boundary either and it is an orbifold according to definition 1.

2. Models for the hyperbolic plane and hyperbolic trigonometry

As we already mentioned, the local geometry of hyperbolic surfaces and of cone-surfaces (outside of cone points) is modeled on the hyperbolic plane $\mathbb{H}^2$. In this section we present two classical models for $\mathbb{H}^2$ and some formulas that can be deduced using these.

One of the standard models is the *upper half-plane* one. Its set of points is

$$\mathbb{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

and the metric is defined by

$$ds = \frac{|dz|}{\mathrm{Im}(z)}.$$

Geodesics are easy to describe in this model: they are either Euclidean vertical half lines or Euclidean half circles orthogonal to the real axis. The group of orientation preserving isometries of $\mathbb{U}$ is $\mathrm{PSL}(2,\mathbb{R})$, acting as Möbius transformation on the upper half-plane:

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{U} \ni (A,z) \mapsto A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \in \mathbb{U}.$$

To generate the full group of isometries we just need to add the map $z \mapsto -\frac{1}{\bar{z}}$.

We can also explicitly give the distance between two points z and w in $\mathbb{U}$:

$$\cosh d(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

It is often useful to consider also the *points at infinity* of the hyperbolic plane, which we can think of as directions of geodesics. In this model, the circle of points at infinity is given by $\mathbb{R} \cup \{\infty\}$.

Another standard model is the *Poincaré disk model*: in this case, the set of points is given by

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

and the metric is

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Here geodesics are diameters of the disk or arcs of Euclidean circles orthogonal to the boundary of $\mathbb{D}$. The orientation preserving isometries are given by maps of the form

$$z \mapsto \frac{az+\bar{c}}{cz+\bar{a}}$$

where a and b are complex numbers with $|a|^2 - |c|^2 = 1$, and we can add the map $z \mapsto \bar{z}$ to generate the full group of isometries.

Again, we have a formula for the distance between two points z and w in $\mathbb{D}$:

$$\sinh^2 \frac{d(z,w)}{2} = \frac{|z-w|}{(1-|z|^2)(1-|w|^2)}.$$

Moreover, the boundary at infinity is the (Euclidean) boundary of the disk, i.e. $\{z \in \mathbb{C} \mid |z| = 1\}$.

The models we introduced allow us to deduce a variety of results about sets in the hyperbolic plane. We mention some of them, the ones which will be fundamental for proofs in the sequel. To lighten the notation, in the remaining of this section we will not distinguish between a segment and its length.

The first facts are about hyperbolic triangles, quadrilaterals and hexagons. They can be found, among many other formulas about hyperbolic polygons, in Chapter 2 of [**Bus10**].

PROPOSITION 2.2. *Let T be a hyperbolic triangle with angles α, β, γ and corresponding opposite sides a, b, c. The following formulas hold.*

(a) $\cosh c = -\sinh a \sinh b \cos\gamma + \cosh a \cosh b$
(b) $\cos\gamma = \sin\alpha \sin\beta \cosh c - \cos\alpha \cos\beta$
(c) $\dfrac{\sinh\alpha}{\sin a} = \dfrac{\sinh\beta}{\sin b} = \dfrac{\sinh\gamma}{\sin c}$.

In particular, we can deduce simpler formulas for right-angled hyperbolic triangles.

PROPOSITION 2.3. *Let T be a hyperbolic triangle as in Proposition 2.2. If γ is a right angle, the following formulas hold.*

(a) $\cosh c = \cosh a \cosh b$
(b) $\cosh c = \cot\alpha \cot\beta$
(c) $\sinh a = \sin\alpha \sinh c$
(d) $\sinh a = \cot\beta \tanh b$
(e) $\cos\alpha = \cosh a \sin\beta$
(f) $\cos\alpha = \tanh b \coth c$.

A *trirectangle* is a quadrilateral with three right angles. Denote by φ the fourth angle, by α and β the two sides adjacent to φ and by a (respectively, b) the side opposite to α (respectively, β).

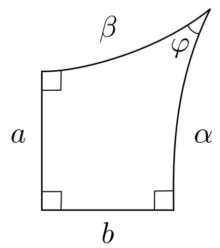


FIGURE 1. A trirectangle

PROPOSITION 2.4. *For a trirectangle as above, the following holds.*

(a) $\cos\varphi = \sinh a \sinh b$
(b) $\cos\varphi = \tanh\alpha \tanh\beta$
(c) $\cosh a = \cosh\alpha \sin\varphi$.

Given a right-angled hexagon, name the sides as in the following picture.

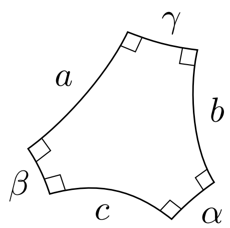


FIGURE 2. A right-angled hexagon

PROPOSITION 2.5. *For a right-angled hexagon as above,*

$$\cosh c = \sinh a \sinh b \cosh\gamma - \cosh a \cosh b.$$

The other result we will need is about hyperbolic disks and can be found in [**Bea83**, Chapter 7].

PROPOSITION 2.6. *(a) The area of a hyperbolic disk of radius r is*

$$2\pi(\cosh r - 1).$$

(b) Given a hyperbolic triangle T of angles α, β and γ, there exists a unique inscribed circle, whose radius R satisfies

$$\tanh^2 R = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1}{2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)}$$

and

$$\tanh R \geq \frac{1}{2}\sin\frac{\text{area}(T)}{2}.$$

The results of Propositions 2.2, 2.3, 2.4 and part *(a)* of Proposition 2.6, can be proved by direct computation in the upper half-plane or the Poincaré disk model. For part *(b)* of Proposition 2.6, we first need to prove that the three bisectors of a triangle meet in a point. Then we can use Proposition 2.3 to deduce the formula and the estimate for the radius.

For more details on what above and for a lot more about the hyperbolic plane, we refer to Beardon's book *The Geometry of Discrete Groups* [**Bea83**], Chapter 7 in particular.

3. Some basic properties of simple closed curves

In the study of hyperbolic structures on surfaces, simple closed curves and simple closed geodesics play a very important role. This section is dedicated to some basic definitions and results, both classical and recent.

A simple closed curve α on a (topological) surface S is called *non-trivial* if it represents a non-trivial element of the fundamental group. It is *essential* if it is non-trivial and no component of $S \setminus \alpha$ is a once-punctured disk. Given two simple closed curves α and β, their *(geometric) intersection number* $i(\alpha, \beta)$ is the minimum number of transverse intersections between a representative of the free homotopy class of α and a representative of the free homotopy class of β. The curves α and β are *in minimal position* if

$$i(\alpha, \beta) = |\alpha \cap \beta|.$$

The *bigon criterion* (see [**FM12**, Chapter 1]) gives a characterization of curves in minimal position. We say that two transverse simple closed curves α and β form a *bigon* if there is an embedded disk in S whose boundary is given by the union of an arc of α and an arc of β, intersecting in exactly two points. The following holds.

Theorem 2.7. (The bigon criterion) *Two transverse simple closed curves α and β are in minimal position if and only if they do not form any bigon.*

In particular, this implies that if two simple closed curves intersect transversally in exactly one point, they are in minimal position (and their intersection number is one).

If we consider curves on cone-surfaces, we will assume that they do not pass through the cone points and the definitions above will be referred to the topological surface $S \setminus X$, where S is the cone-surface and X is the set of cone points.

3.1. Curves on hyperbolic surfaces. This section contains some classical facts about simple closed curves and geodesics on hyperbolic surfaces. Since in the next session we will prove most of the statements in the case of admissible cone-surfaces, here we will just state the results. For proofs, and also for much more about (curves on) hyperbolic surfaces, we refer to Buser's book *Geometry and Spectra of Compact Riemann Surfaces* [**Bus10**]. This book, together with Beardon's [**Bea83**], has been and most likely will be a faithful and very useful companion, and I hope it can be the same for the interested reader.

The first theorem in this section describes the correspondence between homotopy classes of simple closed curves and simple closed geodesics.

THEOREM 2.8. *Let S be a hyperbolic surface.*

(a) Each essential simple closed curve α on S is homotopic to a unique simple closed geodesic, denoted by $\mathcal{G}(\alpha)$.
(b) $\mathcal{G}(\alpha)$ realizes the minimum length among curves in the homotopy class of α.
(c) Given two essential simple closed curves α and β, either $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$ or $|\mathcal{G}(\alpha) \cap \mathcal{G}(\beta)| = i(\alpha, \beta) \leq |\alpha \cap \beta|$.

Note that part *(c)* of Theorem 2.8 tells us that two simple closed geodesics are in minimal position. Indeed, two simple closed geodesics cannot form a bigon (which follows from the fact that two distinct geodesics in the hyperbolic plane cannot form a bigon).

A very useful theorem on simple closed geodesics is the *collar lemma*, which describes the geometry of a neighborhood of a simple closed geodesic (see the papers of Keen [**Kee74**], Randol [**Ran79**] and Buser [**Bus78**], and Buser's book [**Bus10**]). More precisely, given a simple closed geodesic γ we define the associated collar to be

$$\mathscr{C}(\gamma) = \{p \in S \mid d(p, \gamma) < w(\gamma)\},$$

where

$$w(\gamma) = \operatorname{arcsinh} \frac{1}{\sinh \frac{\ell(\gamma)}{2}}.$$

For every cusp c, we denote by $\mathcal{H}_c$ the open horoball region of area 2.

THEOREM 2.9. (COLLAR LEMMA) *Let S be a hyperbolic surface of signature (g, n). Let $c_1, \ldots, c_n$ be the cusps of S and $\{\gamma_1, \ldots, \gamma_k\}$ be a set of pairwise disjoint simple closed geodesics. Then:*

(a) The collars $\mathscr{C}(\gamma_i)$ and the horoballs $\mathcal{H}_{c_j}$ are all disjoint.
(b) Each $\mathscr{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell(\gamma_i)^2 \cosh^2 \rho \, dt^2$.
(c) Each $\mathcal{H}_{c_j}$ is isometric to the quotient $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 1/2\} / \Gamma$, where Γ is the group generated by $T : z \mapsto z + 1$ acting on the upper half-plane by isometries.

Note that since the collars are embedded cylinders, if α and β are simple closed geodesics with $i(\alpha, \beta) = n$, then $\ell(\beta) \geq 2nw(\alpha)$. This means that if one of the two is very short, the other must be very long. In particular, all simple closed geodesics of length less than $2 \operatorname{arcsinh} 1$ are pairwise disjoint.

We will also be interested in *pants decompositions*, i.e. maximal sets of disjoint simple closed geodesics, and *pairs of pants*, i.e. hyperbolic surfaces of signature $(0, 3)$. Topologically, a pair of pants is simply a three-holed sphere.

Often, a pair of pants is defined to be a (topological) three-holed sphere, without any request on the geometry. We use the more restrictive definition above, since it will simplify some statements. If we want to consider the topological version, we will talk about *topological pair of pants.* A *topological pants decomposition* will be a maximal set of pairwise disjoint and non-homotopic essential simple closed curves. Note that by Theorem 2.8, to each topological pair of pants (respectively, topological pants decomposition) we can associate a unique pair of pants (respectively, pants decomposition).

We have the following.

PROPOSITION 2.10. *Let S be a hyperbolic surface of signature (g, n) and $\gamma_1, \ldots, \gamma_k$ be pairwise disjoint simple closed geodesics. Then*

$$k \leq 3g - 3 + n$$

and there exist simple closed geodesics $\gamma_{k+1}, \ldots, \gamma_{3g-3+n}$ such that

$$\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$$

is a pants decomposition. Moreover, each pants decomposition decomposes S into a union of $2g - 2 + n$ pairs of pants.

3.2. Curves on admissible cone-surfaces. In the case of admissible cone-surfaces, we can generalize some of the results of Section 3.1. For this, we need to consider a more general type of curves.

An *admissible geodesic of the first type* is a simple closed geodesic. An *admissible geodesic of the second type* is a curve obtained by following back and forth a simple geodesic between two cone points of order two. Note that if we cut open along an admissible geodesic of the second type we obtain a boundary component which is a simple closed geodesic.

Let X be the set of all cone points of an admissible cone-surface S. Given an admissible geodesic of the second type δ, we define δ_ε to be the boundary of an ε-neighborhood of δ, for $\varepsilon > 0$ small enough so that δ_ε is a simple closed curve, contractible in $(S \setminus X) \cup \delta$. We say that a simple closed curve γ is homotopic to δ if it is homotopic to δ_ε on $S \setminus X$.

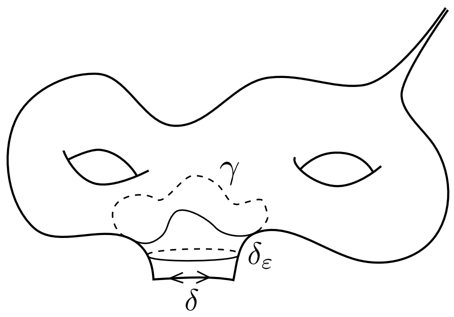


FIGURE 3. An admissible geodesic of the second type and a simple closed curve homotopic to it

The following theorem, which generalizes Theorem 2.8, gives a correspondence between homotopy classes of simple closed curves and admissible geodesics.

THEOREM 2.11. *Let S be an admissible cone-surface and X the set of cone points of S.*

(1) Each essential simple closed curve α on $S \setminus X$ is homotopic to a unique admissible geodesic, denoted $\mathcal{G}(\alpha)$.
(2) $\mathcal{G}(\alpha)$ realizes the minimum length among curves homotopic to α.
(3) Given two essential simple closed curves α and β on $S \setminus X$, either $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$, or $|\mathcal{G}(\alpha) \cap \mathcal{G}(\beta)| \leq |\alpha \cap \beta|$.

Note that in the case of admissible cone-surfaces without cone points of order two, there is no need to introduce admissible geodesics of second type and the correspondence is again between homotopy classes of essential simple closed curves and simple closed geodesics, as in the classical case. This has been proven by Dryden and Parlier in [**DP07**] (for compact surfaces, but their techniques extend to the non compact case as well).

Moreover, part (1) of Theorem 2.11 has been stated in a slightly different form in by Tan, Wong and Zhang in [**TWZ06**]. We will give here an alternative and more detailed proof.

PROOF OF THEOREM 2.11. We start by proving parts (1) and (2). Suppose first that α is homotopic to an admissible geodesic of the second type $\mathcal{G}(\alpha)$.

We begin by proving that

$$\inf\{\ell(\alpha') \mid \alpha' \text{ is homotopic to } \alpha\} = \ell(\mathcal{G}(\alpha)).$$

Let α_ε be the curve obtained as the boundary of an ε-neighborhood of $\mathcal{G}(\alpha)$.

Consider α' homotopic to α. If it is not simple, by a result of Hass and Scott [**HS85**] it forms a bigon or a 1-gon (a disk bounded by an arc of α' starting and ending at the same point and with no other self-intersection), so we can shorten it. Thus we can assume that α' is a simple closed curve. If it crosses $\mathcal{G}(\alpha)$, then for all $\varepsilon > 0$ small enough α' is homotopic to α_ε and α' and α_ε intersect transversally. But α' and α_ε are homotopic, so their intersection number is zero. This implies that they are not in minimal position and, by the bigon criterion, they form a bigon. As a consequence, an arc of α' and an arc of $\mathcal{G}(\alpha)$ bound a disk; as $\mathcal{G}(\alpha)$ is geodesic, this means that we can shorten α'. We can then assume that α' is a simple closed curve which doesn't cross $\mathcal{G}(\alpha)$, so α' determines a curve on the cone-surface obtained by cutting open along $\mathcal{G}(\alpha)$, and by standard hyperbolic geometry $\ell(\alpha') \geq \ell(\mathcal{G}(\alpha))$. So

$$\inf\{\ell(\alpha') \,|\, \alpha' \text{ is homotopic to } \alpha\} \geq \ell(\mathcal{G}(\alpha)),$$

and we have equality since $\ell(\alpha_\varepsilon)$ tends to $\ell(\mathcal{G}(\alpha))$ for ε going to zero.

We still need to show that $\mathcal{G}(\alpha)$ is unique, i.e. that α is not homotopic to any other admissible geodesic. By contradiction, suppose α is homotopic to another admissible geodesic γ. If we cut along γ and $\mathcal{G}(\alpha)$ we get a subset of a hyperbolic cylinder with two simple closed geodesics as boundary. This is impossible, so $\mathcal{G}(\alpha)$ is unique.

We can now assume that α is not homotopic to any admissible geodesic of the second type. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of curves homotopic to α, $\alpha_n \subset S \setminus X$, such that $\ell(\alpha_n)$ converges to $\inf\{\ell(\alpha')) \,|\, \alpha' \text{ is homotopic to } \alpha\}$. Since their length is converging, there is a open subset C of S, given by the union of small open neighborhoods of the cusps, such that the curves α_n are contained in the compact subset $S \setminus C$. If we parametrize all the curves on the same interval with constant speed, by Arzelà–Ascoli Theorem (see for instance [**Bus10**, Appendix]) we get (up to passing to a subsequence) a limit curve $\mathcal{G}(\alpha)$ in $S \setminus C$, homotopic to α, such that

$$\ell(\mathcal{G}(\alpha)) = \inf\{\ell(\alpha')) \,|\, \alpha' \text{ is homotopic to } \alpha\}.$$

Since the curve is length minimizing, it is simple (again by Hass and Scott's result in [**HS85**]) and geodesic outside the cone points. Suppose now that $\mathcal{G}(\alpha)$ passes through a cone point p. Then both angles that $\mathcal{G}(\alpha)$ forms at p are strictly less than π, so it can be shortened while staying in the same homotopy class. This is a contradiction to the minimality of $\ell(\mathcal{G}(\alpha))$. Hence $\mathcal{G}(\alpha)$ is a simple closed geodesic. If there is another simple closed geodesic γ homotopic to α, cut along γ and $\mathcal{G}(\alpha)$. We obtain a hyperbolic cylinder with two distinct simple closed geodesics as boundary, a contradiction. Hence the uniquess of $\mathcal{G}(\alpha)$.

We now show part (3). Assume $G(\alpha)$ and $\mathcal{G}(\beta)$ are distinct. Since they are geodesics, they do not form any bigons.

Suppose $\mathcal{G}(\alpha)$ and $\mathcal{G}(\beta)$ are simple closed geodesics. By the bigon criterion they intersect minimally and

$$|\mathcal{G}(\alpha) \cap \mathcal{G}(\beta)| = i([\alpha], [\beta]) \leq |\alpha \cap \beta|.$$

If $\mathcal{G}(\alpha)$ is an admissible geodesic of the second type and $\mathcal{G}(\beta)$ is a simple closed geodesics (or vice versa), we can choose a curve $\alpha' \in [\alpha]$ in a small neighborhood of $\mathcal{G}(\alpha)$ such that it doesn't form any bigons with $\mathcal{G}(\beta)$. So

$$|\alpha' \cap \mathcal{G}(\beta)| = i([\alpha], [\beta]) \leq |\alpha \cap \beta|.$$

Since every intersection of $\mathcal{G}(\alpha)$ and $\mathcal{G}(\beta)$ corresponds to at least two intersections of α' and $\mathcal{G}(\beta)$, we have

$$|\mathcal{G}(\alpha) \cap \mathcal{G}(\beta)| \leq |\alpha' \cap \mathcal{G}(\beta)| \leq |\alpha \cap \beta|.$$

If $G(\alpha)$ and $\mathcal{G}(\beta)$ are admissible geodesics of the second type, we consider

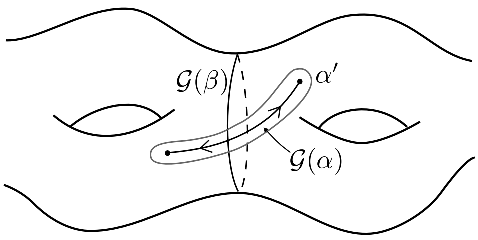


FIGURE 4. $\mathcal{G}(\alpha)$, $\mathcal{G}(\beta)$ and the curve α'

two curves α' and β' in small neighborhoods of the geodesics and we apply a similar argument to the one above. □

Also for cone-surfaces we will be interested in decompositions into three-holed spheres; we define a *(generalized) pants decomposition* to be a maximal set of pairwise disjoint admissible geodesics. A *(generalized) pair of pants* is an admissible cone-surface of signature $(0, n, b)$, with $n + b = 3$. A pair of pants will be called:

- a *Y-piece* if it has three simple closed curves as boundary,
- a *V-piece* if it has two simple closed curves and a singular point as boundary,
- a *joker's hat* if it has two singular points and a simple closed curve as boundary,
- a *triangular surface* if it has three singular points as boundary.

The corresponding result to Proposition 2.10 is the following.

PROPOSITION 2.12. *Let S be an admissible cone-surface of signature (g, n, b) and $\gamma_1, \ldots, \gamma_k$ be pairwise disjoint admissible geodesics. Then*

$$k \leq 3g - 3 + n + b$$

and there exist admissible geodesics $\gamma_{k+1}, \ldots, \gamma_{3g-3+n+b}$ such that

$$\{\gamma_1, \ldots, \gamma_{3g-3+n+b}\}$$

is a pants decomposition. Moreover, each pants decomposition decomposes S into a union of pairs of pants.

PROOF. Let X be the set of cone points of S. We associate to $\gamma_1, \ldots, \gamma_k$ a set $\{\gamma_1', \ldots \gamma_k'\}$ of pairwise disjoint simple closed curves on $S \setminus X$ such that $\mathcal{G}(\gamma_i') = \gamma_i$ for every i. Then the γ_i's are pairwise non-homotopic on $S \setminus X$. Since $S \setminus X$ is homeomorphic to a surface of genus g with $n + b$ points removed, $k \leq 3g - 3 + n + b$ and there exist simple closed curves $\gamma_{k+1}', \ldots, \gamma_{3g-3+n+b}'$ on $S \setminus X$, disjoint and not homotopic to any γ_i' such that $\{\gamma_1', \ldots, \gamma_{3g-3+n+b}'\}$ is a topological pants decomposition of $S \setminus X$. Let $\gamma_j = \mathcal{G}(\gamma_j')$ for every $j \in \{k+1, \ldots, 3g - 3 + n + b\}$. By Proposition 2.11, the set $\{\gamma_1, \ldots, \gamma_{3g-3+n+b}\}$ is a pants decomposition of S. Moreover, since $\{\gamma_1', \ldots, \gamma_{3g-3+n+b}'\}$ decomposes $S \setminus X$ into topological pairs of pants, $\{\gamma_1, \ldots, \gamma_{3g-3+n+b}\}$ decomposes S into pairs of pants. □

Note that in the case of admissible cone-surfaces, the number of pairs of pants obtained by cutting along a pants decomposition is not determined by the signature. For example, consider a sphere with four cone points, two of order 2 and two of order 3. We can choose two different curves, each forming a pants decomposition; one decomposes the orbifold into two pairs of pants and the other into one, as in the following picture:

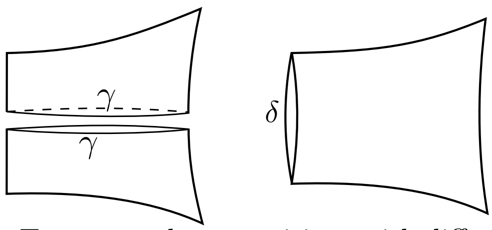


FIGURE 5. Two pants decompositions with different number of pairs of pants

We would also like to have a corresponding result to Theorem 2.9. Unfortunately, as remarked by Dryden and Parlier in [**DP07**], this cannot be generalized to all admissible cone-surfaces, as two cone points of order two can be arbitrarily close to each other or to another simple closed geodesic. On the other hand, Dryden and Parlier show that one can generalize this result in the case of cone-surfaces where all cone points have total angle strictly less than π. They prove the following:

THEOREM 2.13. *Let S be a compact cone-surface of genus g with n cone-points $p_1, \dots, p_n$ with total angles $2\varphi_1, \dots, 2\varphi_n$. Let 2φ be the largest total angle, and assume $2\varphi < \pi$. Let $\gamma_1, \dots, \gamma_k$ be disjoint simple closed geodesics on S. Then the following hold:*

(a) The collars

$$\mathcal{C}(\gamma_i) = \left\{ p \in S \,|\, d(p, \gamma_k) \leq a_i = \operatorname{arcsinh} \frac{\cos\varphi}{\sinh\frac{\ell(\gamma_k)}{2}} \right\}$$

and

$$\mathcal{C}(p_j) = \left\{ p \in S \,|\, d(p, p_j) \leq b_j = \operatorname{arccosh} \frac{1}{\sin\varphi_j} \right\}$$

are pairwise disjoint.

(b) Each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-a_i, a_i] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell(\gamma_i)^2 \cosh^2\rho\, dt^2$.

(c) Each $\mathcal{C}(p_j)$ is isometric to the hyperbolic cone $[0, b_j] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + \frac{\varphi_j^2}{\pi^2} \sinh^2\rho\, dt^2$.

CHAPTER 3

Maximum injectivity radius of orbifolds

Among the objects that give us information on hyperbolic structures on surfaces, we are especially interested in embedded disks. In particular, we can ask the following: given a hyperbolic orbifold, how big can an isometrically embedded disk be? Is there a disk we can embed in any orbifold?

It is quite easy to see that the answer to the second question is yes. Indeed, one can show that there is some positive constant A such that each orbifold contains an embedded triangle of area at least A. Then by Proposition 2.6, this triangle, and hence any orbifold, contains an embedded disk of some radius r depending only on A. What is much more challenging is to actually compute a sharp lower bound and to show which orbifolds realize the bound; this problem is the subject of this chapter. We start with the necessary definitions to state precisely the question we are interested in.

Let S be an admissible cone-surface; for every point $p \in S$, the *injectivity radius* at p is

$$r_p := \max\{r \geq 0 \,|\, D_r(p) \text{ is isometric to an open disk of radius } r \text{ in } \mathbb{H}^2\}.$$

We define the map

$$\begin{aligned} r : S &\to \mathbb{R} \\ p &\mapsto r_p. \end{aligned}$$

LEMMA 3.1. *The map r is continuous and admits a global maximum.*

PROOF. Let $\varepsilon > 0$ be small. If $d(p,q) < \varepsilon$, then $r_q \geq r_p - \varepsilon$, because $D_{r_p-\varepsilon}(q)$ is embedded in $D_{r_p}(p)$; conversely $r_p \geq r_q - \varepsilon$. So $|r_p - r_q| \leq \varepsilon$ and the map r is continuous.

If there is no cusp, S is compact and r has a maximum. If there are any cusps, note that the injectivity radius becomes smaller while getting closer to the cusp. So

$$\sup_{p \in S} r_p = \sup_{p \in C} r_p$$

for some compact set C in S obtained by taking away suitable open horoballs. Again, we get that r has a maximum. $\square$

We define the *maximum injectivity radius* of an admissible cone-surface S to be

$$r(S) := \max_{p \in S} r_p.$$

The problem we are interested in is finding a sharp lower bound for the maximum injectivity radius of all orbifolds.

We will first work with triangular surfaces, for which we will be able to actually give an implicit formula for the maximum injectivity radius as a function of the cone angles. We will then show that the proper lower bound is a lower bound for all orbifolds.

1. Triangular surfaces

Recall that a triangular surface is an admissible cone-surface of signature $(0,3)$. Every triangular surface can be obtained by gluing two hyperbolic triangles of angles α, β and γ, all less or equal to $\frac{\pi}{2}$; we denote the corresponding admissible cone-surface by $S_{\alpha,\beta,\gamma}$. We call A, B and C the vertices of the triangles corresponding to α, β and γ respectively and a, b and c the opposite sides (or the lengths of the sides). Such a decomposition in triangles in unique. Moreover, every triangle is uniquely determined (up to isometry) by its angles. So the moduli space of triangular surfaces is

$$\mathcal{M} = \left\{ (\alpha_1, \alpha_2, \alpha_3) \in \left[0, \tfrac{\pi}{2}\right]^3 \mid \textstyle\sum_{i=1}^3 \alpha_i < \pi \right\} \Big/ \mathrm{Sym}(3),$$

where the action of the symmetric group $\mathrm{Sym}(3)$ on the set of triples is given by

$$(\sigma, (\alpha_1, \alpha_2, \alpha_3)) \mapsto (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}).$$

When dealing with surfaces with cusps, it is often useful to consider horoball regions associated to the cusps. The first result we need is about how big a horoball embedded in a triangular surface can be, in terms of the cone angles.

LEMMA 3.2. *If a triangular surface S contains a cusp and two singular points with total angles $2\theta_1$ and $2\theta_2$, the associated horocycle of length $h(S)$ is embedded in S, where*

$$h(S) := \frac{4}{\sqrt{1 + \frac{1}{R(0,\theta_1,\theta_2)}}}$$

and

$$R(0, \theta_1, \theta_2) = \operatorname{arctanh} \frac{\cos\theta_1 + \cos\theta_2}{2\sqrt{(1+\cos\theta_1)(1+\cos\theta_2)}}$$

is the radius of the inscribed disk in a triangle with angles 0, θ_1 and θ_2.

PROOF. Consider T, one of the two triangles that form S, and let p be the center of the inscribed disk in T. Consider the horocycle passing through the points on b and c at distance $R(0, \theta_1, \theta_2)$ from p. This is an embedded horocycle and direct computation shows that it is longer than $h(S)$. □

NOTATION: if A (respectively, B, C) is a cusp, we denote h_A (respectively, h_B, h_C) the associated horocycle of length $h(S)$.

1.1. The maximum injectivity radius of triangular surfaces. Let us fix the angles α, β and γ and denote $S_{\alpha,\beta,\gamma}$ simply by S. For any point p on the surface, we have a maximal embedded disk centered at p, i.e. $D_{r_p}(p)$. The maximality implies that either a cone point of order two belongs to $\partial D_{r_p}(S)$, or there is a point p on the boundary and two distinct radii from p to q. In the second case, we say that the disk is tangent to itself in q.

For every point where $D_{r_p}(p)$ is tangent to itself, the two radii form a simple loop, which is geodesic except in p. Its length is $2r_p$ and it is length-minimizing in its class in $\pi_1(S \setminus \{A, B, C\}, p)$ (otherwise the disk would be overlapping at q).

If there is a cone point of order two, say A, and it belongs to the boundary of $D_{r_p}(p)$, we associate a loop obtained by traveling from p to A and back on the length realizing geodesic between these two points. Clearly, this loop has length $2r_p$.

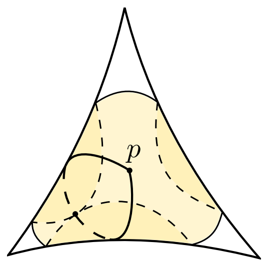


FIGURE 1. A tangency point and its associated loop

Now, let us forget disks for some time and concentrate on defining and studying loops of the type described just above. Fix a non-singular point p and consider one corner, say A. We define the associated loop γ_A to be:

- if A is a cone point of order two, the curve that traces the length realizing geodesic between p and A from p to A and back;
- otherwise, the simple loop based at p, going around A, and geodesic except at p.

We need to show that γ_A is well defined, i.e. that there exists a unique associated loop (if A is not a cone point of order two, otherwise it is already clear). This is proven in the following lemma.

LEMMA 3.3. *Suppose A is not a cone point of order two. Then there exists a unique simple loop γ_A based at p, going around A, geodesic except at p. Moreover, it is length minimizing in its class in $\pi_1(S \setminus \{A, B, C\}, p)$.*

PROOF. Consider the class $\mathcal{C} \in \pi_1(S \setminus \{A, B, C\}, p)$ of a simple closed curve with base point p that goes around A. We first show the uniqueness of γ_A. Suppose then γ_A is a simple loop in $\mathcal{C}$, geodesic except at p. Then it doesn't cross the length-realizing geodesic between p and A, otherwise they would form a geodesic bigon. We can cut along the geodesic; γ_A is given by the unique geodesic between the two copies of p on the cut surface.

We show now that γ_A exists and is length-minimizing. Let γ_n be a sequence of smooth curves in $\mathcal{C}$ with lengths converging to $\inf\{\ell(\gamma) \mid \gamma \in \mathcal{C}\}$. If there is any cusp on S, since the length of the curves γ_n is bounded above, we can assume that they are contained in a compact subset of S. By the Arzelà–Ascoli Theorem (see [**Bus10**, Theorem A.19]) we get a limit curve γ_A. Note that γ_A doesn't contain A: it is clear if A is a cusp, while if $\alpha > 0$ and γ_A passes through A, it forms an angle at A smaller or equal to $2\alpha < \pi$, so it can be shortened to a curve in $\mathcal{C}$, contradiction. In particular, $\gamma_A \in \mathcal{C}$. By the minimality, γ_A is simple and geodesic except at p. □

In the next lemma we show that γ_A is length minimizing in its class also if A is a cone point of order two.

LEMMA 3.4. *Suppose A is a cone point of order two and let $\mathcal{C}$ be the class in $\pi_1(S \setminus \{A, B, C\}, p)$ of a simple loop around A and based at p. Then*

$$\ell(\gamma_A) = \inf\{\ell(\gamma) | \gamma \in \mathcal{C}\}.$$

PROOF. Consider $\gamma \in \mathcal{C}$. If it crosses γ_A, the two curves form a bigon and γ can be shortened while staying in the same homotopy class. So, to compute $\inf\{\ell(\gamma) \mid \gamma \in \mathcal{C}\}$ we can consider curves γ which do not cross γ_A. We cut along γ and γ_A and we get a subset of $\mathbb{H}^2$ bounded by two curves between two copies of P: the first one is a geodesic given by γ_A and the second one is given by γ, hence $\ell(\gamma) \geq \ell(\gamma_A)$. One can construct curves in the class with length arbitrarily close to $\ell(\gamma_A)$, so the infimum is $\ell(\gamma_A)$. □

NOTATION: $\ell_A := \ell(\gamma_A)$, $\ell_B := \ell(\gamma_B)$ and $\ell_C := \ell(\gamma_C)$. By $\tilde{\alpha}$ (respectively, $\tilde{\beta}$, $\tilde{\gamma}$) we denote the acute angle of γ_A (respectively, γ_B, γ_C) at p.

Clearly, for $\alpha = \frac{\pi}{2}$, the length is twice the distance $d(p, A)$. So ℓ_A increases (continuously) when $d(p, A)$ increases.

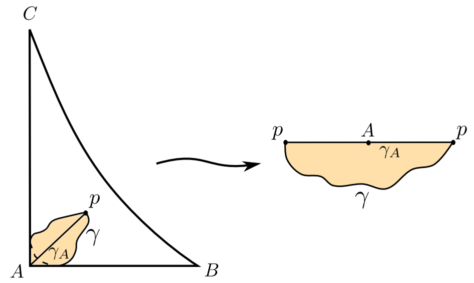


FIGURE 2. Cutting along γ_A for A cone point of order two

If $\alpha = 0$, i.e. if A is a cusp, consider the horocycle h_A. Cut along the loop itself and the geodesic from p to the cusp. We represent the triangle we obtain in the upper half plane model of $\mathbb{H}^2$, choosing A to be the point at infinity. Note that the geodesic between p and the cusp is the bisector of $\tilde{\alpha}$. By Proposition 2.3, we get

$$1 = \cosh \frac{\ell_A}{2} \sin \frac{\tilde{\alpha}}{2}.$$

Let $d_{h_A}(p)$ be $d(p, h_A)$, if p doesn't belong to the horoball bounded by h_A, and $-d(p, h_A)$ otherwise. By direct computation we get that ℓ_A is a continuous monotone decreasing function of $\tilde{\alpha}$, so a continuous monotone increasing function of $d_{h_A}(p)$. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, we cut along the geodesic from p to A and

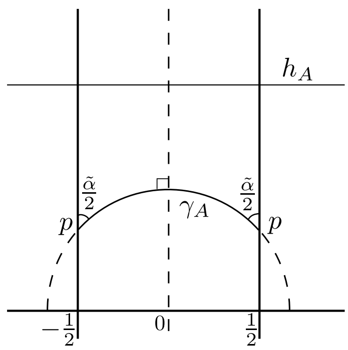


FIGURE 3. The loop around a cusp in the upper half plane

we get an isosceles triangle. Again, the geodesic from p to A is a bisector of $\tilde{\alpha}$. By Proposition 2.3, we obtain the equations

$$\cos \alpha = \cosh \frac{\ell_A}{2} \sin \frac{\tilde{\alpha}}{2}$$

and

$$\sinh \frac{\ell_A}{2} = \sin \alpha \sinh d(p, A).$$

In particular, ℓ_A is a continuous monotone increasing function of $d(p, A)$.

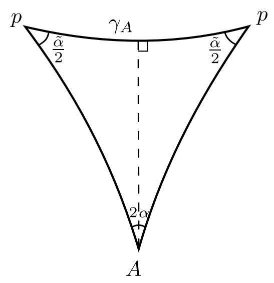


FIGURE 4. The triangle associated to the loop for $\alpha \in \left(0, \frac{\pi}{2}\right)$

So, for $p \in S$, we can define

$$\operatorname{dist}_A(p) = \begin{cases} d(p, A) \text{ if } \alpha \neq 0 \\ d_{h_A}(p) \text{ if } \alpha = 0 \end{cases}$$

Similarly for B and C. We have just proven the following:

LEMMA 3.5. *The length ℓ_A (resp. ℓ_B, ℓ_C) is a continuous monotone increasing function of* $\operatorname{dist}_A(p)$ *(resp.* $\operatorname{dist}_B(p)$, $\operatorname{dist}_C(p)$*).*

Now, given a point p on the surface, we can cut along the loops γ_A, γ_B and γ_C. If there is no cone point of order two, we obtain four pieces: three of them are associated each to a singular point, and the fourth one is a triangle of side lengths ℓ_A, ℓ_B and ℓ_C.

If there is a cone point of order 2, we get only three pieces: two associated to the other singular points and again a triangle of side lengths ℓ_A, ℓ_B and ℓ_C.

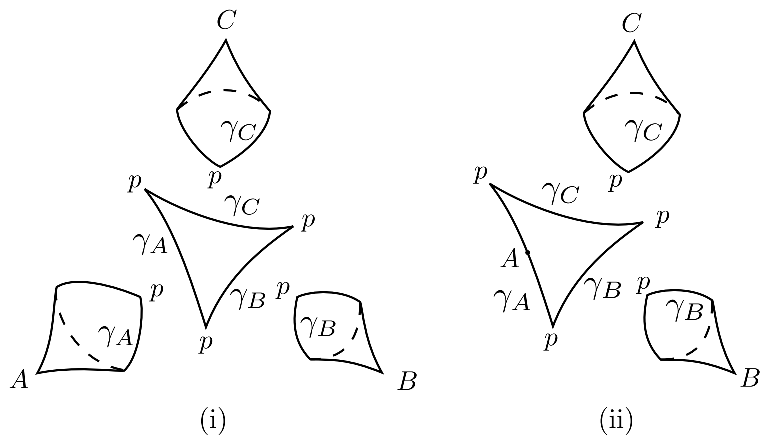


FIGURE 5. S cut along the three loops: (i) without cone points of order two, (ii) with A a cone point of order two

We now want to relate the lengths ℓ_A, ℓ_B and ℓ_C with the injectivity radius in a point. We first need two lemmas.

LEMMA 3.6. *Given a triangular surface S, a non-singular point $p \in S$ and its three associated loops, we have*

$$r_p = \min\left\{\frac{\ell_A}{2}, \frac{\ell_B}{2}, \frac{\ell_C}{2}\right\}.$$

PROOF. Denote $m := \min\left\{\frac{\ell_A}{2}, \frac{\ell_B}{2}, \frac{\ell_C}{2}\right\}$.

We already remarked that either $D_{r_p}(p)$ is tangent to itself in at least a point or its boundary contains a cone point of order 2. In both situations, we get a loop at p of length $2r_p$. So $r_p \geq m$.

To prove the other inequality, suppose γ_A is the shortest of the three loops, i.e. $\ell_A = 2m$. If $\alpha = \frac{\pi}{2}$, then $r_p \leq d(p, A) = \frac{\ell_A}{2} = m$, since an embedded disk cannot contain a singular point. Otherwise, if $\alpha < \frac{\pi}{2}$, consider the point q on γ_A at distance $\frac{\ell_A}{2} = m$ from p. Every disk of radius bigger than $\frac{\ell_A}{2}$ overlaps in q, hence again $r_p \leq m$. □

Let $T \subseteq \mathbb{H}^2$ be a hyperbolic triangle with angles α, β, γ at most $\frac{\pi}{2}$. If $\alpha = 0$, consider a horocycle H_A based at A such that

$$\ell(H_A \cap T) = \frac{1}{2}h(\alpha, \beta, \gamma).$$

For $p \in \mathbb{H}^2$, we define $\mathrm{dist}_A(p)$, $\mathrm{dist}_B(p)$ and $\mathrm{dist}_C(p)$ as before in the case of a triangular surface, with H_A instead of h_A.

LEMMA 3.7. *Let $T \subseteq \mathbb{H}^2$ be a hyperbolic triangle and $p, p' \in T$. If*

$$\begin{cases} \mathrm{dist}_A(p') \geq \mathrm{dist}_A(p) \\ \mathrm{dist}_B(p') \geq \mathrm{dist}_B(p) \\ \mathrm{dist}_C(p') \geq \mathrm{dist}_C(p) \end{cases}$$

then $p = p'$.

PROOF. Consider a corner, say A (similarly for B and C): we define

$$\mathcal{C}_A := \{q \in \mathbb{H}^2 \mid \mathrm{dist}_A(q) = \mathrm{dist}_A(p)\}$$
$$\mathcal{D}_A := \{q \in \mathbb{H}^2 \mid \mathrm{dist}_A(q) \leq \mathrm{dist}_A(p)\}.$$

So $p \in \mathcal{C}_A \cap \mathcal{C}_B \cap \mathcal{C}_C \subseteq \mathcal{D}_A \cap \mathcal{D}_B \cap \mathcal{D}_C$.

Note that if A is not an ideal point, $\mathcal{C}_A$ is a circle and $\mathcal{D}_A$ is the disk bounded by $\mathcal{C}_A$. If A is an ideal point, $\mathcal{C}_A$ the horocycle based at A passing through p and $\mathcal{D}_A$ is the associated horoball. In any case, $\mathcal{D}_A$ is a convex set.

Let $\mathcal{D}$ be the union $\mathcal{D}_A \cup \mathcal{D}_B \cup \mathcal{D}_C$. Since $\mathcal{D}$ is star-shaped with respect to p and it contains the three sides, $T \subseteq \mathcal{D}$.

Consider now $p' \in T$ with

$$\begin{cases} \mathrm{dist}_A(p') \geq \mathrm{dist}_A(p) \\ \mathrm{dist}_B(p') \geq \mathrm{dist}_B(p) \\ \mathrm{dist}_C(p') \geq \mathrm{dist}_C(p) \end{cases}$$

Then

$$p' \in T \setminus (\mathring{\mathcal{D}}_A \cup \mathring{\mathcal{D}}_B \cup \mathring{\mathcal{D}}_C).$$

Since $T \subseteq \mathcal{D}$ we have $p' \in \mathcal{C}_A \cap \mathcal{C}_B \cap \mathcal{C}_C$. We show that $\mathcal{C}_A \cap \mathcal{C}_B \cap \mathcal{C}_C$ is only one point, hence $p = p'$. Suppose by contradiction that the intersection contains two points. In the Poincaré disk model, $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$ are Euclidean circles, so if they intersect in two points, these should both belong to the boundary of $\mathcal{D}$. In particular $p \in T \cap \partial\mathcal{D} \subseteq \partial T$. Suppose without loss of generality $p \in c$; then the circles $\mathcal{C}_A$ and $\mathcal{C}_B$ are tangent in p, so they intersect only in one point. As a consequence, $\mathcal{C}_A \cap \mathcal{C}_B \cap \mathcal{C}_C$ can contain at most one point, a contradiction. □

We are finally ready to characterize points with maximum injectivity radius.

PROPOSITION 3.8. *Given a point $p \in S$, the following are equivalent:*

(1) p is a global maximum point for the injectivity radius,
(2) p is a local maximum point for the injectivity radius,
(3) the three loops at p have the same length $2r_p$.

Moreover, there are exactly two global maximum points.

PROOF. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) By contradiction, suppose only one or two loops have length $2r_p$. Without loss of generality, assume $\ell_A \leq \ell_B \leq \ell_C$; by Lemma 3.6, $r_p = \frac{\ell_A}{2}$.

If only one loop has length $2r_p$, then $\ell_A < \ell_B \leq \ell_C$. Let us consider the geodesic segment between p and A. Every point p' on the prolongation of the geodesic segment after p satisfies $\mathrm{dist}_A(p') > \mathrm{dist}_A(p)$. Thus every such p' has a longer loop around A. By continuity of the lengths of the loops, for points that are sufficiently close to p the loop around A is still the shortest. These points have injectivity radius bigger than r_p, i.e. p is not a local maximum point.

If two loops have length $2r_p$, we have $\ell_A = \ell_B < \ell_C$; we choose this time the orthogonal l from p to the side c. Every point p' on l that is further away from c then p satisfies $\mathrm{dist}_A(p') > \mathrm{dist}_A(p)$ and $\mathrm{dist}_B(p') > \mathrm{dist}_B(p)$.

So like before, every point on l in a small enough neighborhood of p has bigger injectivity radius than r_p and again p is not a local maximum point.

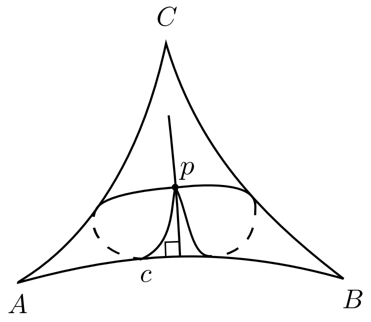


FIGURE 6. P with the orthogonal to c

(3) $\Rightarrow$ (1) Suppose $\ell_A = \ell_B = \ell_C = 2r_p$ and suppose by contradiction that p isn't a global maximum point. Then there exists q with $r_q > r_p$. Let ψ be the orientation reversing isometry that exchanges the two triangles forming S and fixes the sides pointwise. By possibly replacing q with $\psi(q)$, we can assume q belongs to the same triangle as p. Since $r_q > r_p$, the loops at q are strictly longer than the ones at p. So $\text{dist}_A(q) > \text{dist}_A(p)$, $\text{dist}_B(q) > \text{dist}_B(p)$ and $\text{dist}_C(q) > \text{dist}_C(p)$, in contradiction with Lemma 3.7.

We still have to show that there are exactly two global maximum points. With similar arguments as the ones above, if two points p and p' on one triangle are global maximum points for the injectivity radius, then $p = p'$. So we have at most one global maximum point per triangle. To conclude that there are exactly two global maximum points, it is enough to show that a global maximum point cannot belong to a side. Suppose it does and assume, without loss of generality, $p \in c$. Since the geodesics from A to p and from B to p are bisectors of $\tilde{\alpha}$ and $\tilde{\beta}$ respectively, the loops γ_A and γ_B meet the sides b and a orthogonally. So $d(p, a) = d(p, b) = r_P$. Let q and q' be the intersections of γ_C with a and b. Then

$$2r_p = \ell_C > d(p, p) + d(p, q') \geq d(p, a) + d(p, b) = 2r_p,$$

a contradiction. □

Now that we have established a criterion to detect points with maximum injectivity radius, we want to transform it into a system of equations, which will give us an implicit formula for $r(S)$.

Suppose first $\alpha, \beta, \gamma < \frac{\pi}{2}$, and let p be a maximum for the injectivity radius. By Proposition 3.8, the three loops γ_A, γ_B and γ_C have length $2r_p$. Cutting along the three loops we get four pieces; the three associated to the singular points give us the equations

$$\cos\alpha = \cosh r_p \sin\frac{\tilde{\alpha}}{2}$$
$$\cos\beta = \cosh r_p \sin\frac{\tilde{\beta}}{2}$$
$$\cos\gamma = \cosh r_p \sin\frac{\tilde{\gamma}}{2}.$$

Vice versa, any positive solution to these equations determines the three pieces.

The fourth piece is a triangle, equilateral since the three sides are the three loops. Denote by θ the angle of the triangle. By hyperbolic trigonometry we have

$$\cosh 2r_p = \frac{\cos\theta + \cos^2\theta}{\sin^2\theta},$$

or equivalently

$$\cosh r_p = \sqrt{\frac{1}{2(1-\cos\theta)}}.$$

A positive solution of the above equation determines the triangle. Note that $\theta \neq 0$, because otherwise two loops would have the same direction in p, which is impossible.

Since p is not a singular point, the angles at it should sum up to 2π. Thus, we obtain the following system:

$$(1) \qquad \begin{cases} \cos\alpha = \cosh r_p \sin\frac{\tilde{\alpha}}{2} \\ \cos\beta = \cosh r_p \sin\frac{\tilde{\beta}}{2} \\ \cos\gamma = \cosh r_p \sin\frac{\tilde{\gamma}}{2} \\ \cosh r_p = \sqrt{\frac{1}{2(1-\cos\theta)}} \\ \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + 3\theta = 2\pi \end{cases}$$

We want a solution that satisfies $r_p > 0$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in (0, \pi)$ and $\theta \in (0, \frac{\pi}{3})$. If we have such a solution to the system, we have four pieces that can be glued to form the surface S.

Under the conditions $r_p > 0$, $\tilde{\alpha}, \tilde{\beta} \in [0, \pi)$ and $\theta \in (0, \frac{\pi}{3})$, the system (1) is equivalent to

$$(2) \qquad \begin{cases} \sin\frac{\tilde{\alpha}}{2} = \frac{\cos\alpha}{\cosh r_p} \\ \sin\frac{\tilde{\beta}}{2} = \frac{\cos\beta}{\cosh r_p} \\ \cos\theta = 1 - \frac{1}{2\cosh^2 r_p} \\ \tilde{\gamma} = 2\pi - \tilde{\alpha} - \tilde{\beta} - 3\theta \\ F_{\alpha,\beta,\gamma}(r_p) = 0 \end{cases}$$

where

$$\begin{aligned} f_{\alpha,\beta,\gamma}(x) = 2\cos\gamma\, x^4 - (3x^2-1)(\sqrt{x^2-\cos^2\alpha}\sqrt{x^2-\cos^2\beta} - \cos\alpha\cos\beta) \\ - (x^2-1)\sqrt{4x^2-1}(\cos\alpha\sqrt{x^2-\cos^2\beta} + \cos\beta\sqrt{x^2-\cos^2\alpha}) \end{aligned}$$

and $F_{\alpha,\beta,\gamma} = f_{\alpha,\beta,\gamma} \circ \cosh$.

When the triple (α,β,γ) is fixed and it is clear to which angles we are referring to, we will simply call f (respectively F) the function $f_{\alpha,\beta,\gamma}$ (respectively $F_{\alpha,\beta,\gamma}$).

REMARK 3.9. *The special role of γ in the function $f_{\alpha,\beta,\gamma}$ follows from the choice to express $\tilde{\gamma}$ in terms of the other angles.*

Suppose now that one angle, say α, is a right angle. Then we have only three pieces obtained by cutting S along the three loops. We have the following system:

$$\begin{cases} \cos\beta = \cosh r_p \sin\frac{\tilde{\beta}}{2} \\ \cos\gamma = \cosh r_p \sin\frac{\tilde{\gamma}}{2} \\ \cosh r_p = \sqrt{\frac{1}{2(1-\cos\theta)}} \\ \tilde{\beta} + \tilde{\gamma} + 3\theta = 2\pi \end{cases} \tag{3}$$

Again, we want a solution that satisfies $r_p > 0$, , $\tilde{\beta}, \tilde{\gamma} \in (0,\pi)$ and $\theta \in (0, \frac{\pi}{3})$.

REMARK 3.10. *If we ask $r_p > 0$, the equation*

$$\cos\alpha = \cosh r_p \sin\frac{\tilde{\alpha}}{2}$$

has $\tilde{\alpha} = 0$ as solution if and only if $\alpha = \frac{\pi}{2}$.

From the previous remark it follows that we can consider the same system (2) for any surface. We look for a solution $r_p > 0$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in [0,\pi)$ and $\theta \in (0, \frac{\pi}{3})$.

PROPOSITION 3.11. *There exists a unique solution to (2) satisfying $r_p > 0$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in [0,\pi)$ and $\theta \in (0, \frac{\pi}{3})$.*

PROOF. One can check that $f(1) > 0$ and $\lim_{x\to+\infty} f(x) = -\infty$. Since f is continuous, it has a zero in $(1,+\infty)$, i.e. F has a zero in $(0,+\infty)$.

To prove that the whole system has a solution, we have to find non-negative angles satisfying the equations above. Given r_p such that $F(r_p) = 0$, there are unique $\frac{\tilde{\alpha}}{2}, \frac{\tilde{\beta}}{2} \in [0, \pi/2)$ that satisfy

$$\sin \frac{\tilde{\alpha}}{2} = \frac{\cos \alpha}{\cosh r_p}$$
$$\sin \frac{\tilde{\beta}}{2} = \frac{\cos \beta}{\cosh r_p}.$$

Since

$$\cos \theta = 1 - \frac{1}{2\cosh^2 r_p} \in \left(\frac{1}{2}, 1\right),$$

there is a unique solution $\theta \in \left(0, \frac{\pi}{3}\right)$. As

$$\tilde{\gamma} = 2\pi - \tilde{\alpha} - \tilde{\beta} - 3\theta \in (-\pi, 2\pi),$$

we get $\frac{\tilde{\gamma}}{2} \in \left(-\frac{\pi}{2}, \pi\right)$. Using the equivalence with system (1), we know that

$$\sin \frac{\tilde{\gamma}}{2} = \frac{\cos \gamma}{\cosh r_p} \in [0, 1),$$

thus $\tilde{\gamma}/2 \in [0, \pi)$. So, for every zero of F, we have a unique solution $(r_p, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \theta)$ to (2) with the required conditions.

A solution gives us a radius and two associated centers, one for each triangle. We know, by Proposition 3.8, that there exist exactly two maximum points on S, which guarantees unicity of the solution of (2). □

From the proof above we also see that $r(S)$ is the unique positive solution of $F = 0$; for $r \geq 0$

$$F(r) > 0 \Leftrightarrow r < r(S)$$

and

$$F(r) < 0 \Leftrightarrow r > r(S).$$

In terms of f, for $x \geq 1$ we have

$$f(x) > 0 \Leftrightarrow x < \cosh r(S)$$

and

$$f(x) < 0 \Leftrightarrow x > \cosh r(S).$$

As last result of this section, we show that for triangular surfaces, the maximum injectivity radius depends continuously on the angles. To do so, consider the function

$$\mathcal{F} : \mathcal{A} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(\alpha, \beta, \gamma, y) \longmapsto \mathcal{F}(\alpha, \beta, \gamma, y) := F_{\alpha,\beta,\gamma}(y)$$

where $\mathcal{A}$ is the set of possible triples of angles:

$$\mathcal{A} = \left\{(\alpha, \beta, \gamma) \in \left[0, \frac{\pi}{2}\right]^3 : \alpha + \beta + \gamma < \pi\right\}.$$

Note that $\mathcal{F}$ is continuous, as one can see from the explicit form of $F_{\alpha,\beta,\gamma}(y)$.

REMARK 3.12. *Given a triangular surface S and a point $p \in S$ with maximum injectivity radius, we have*

$$\text{area}\left(D_{r(S)}(p)\right) \le \text{area}(S) < 2\pi.$$

So there is a constant $M > 0$ such that $r(S) \le M$ for all triangular surfaces.

PROPOSITION 3.13. *The map*

$$\begin{aligned} r(S_*) &: \mathcal{A} \to \mathbb{R} \\ (\alpha, \beta, \gamma) &\mapsto r(S_{\alpha,\beta,\gamma}) \end{aligned}$$

is continuous.

PROOF. To prove the continuity, we show that for every sequence

$$\{(\alpha_n, \beta_n, \gamma_n)\}_{n=1}^{+\infty} \subseteq \mathcal{A}$$

converging to some triple $(\alpha, \beta, \gamma) \in \mathcal{A}$, the limit $\lim\limits_{n\to+\infty} r(S_{\alpha_n,\beta_n,\gamma_n})$ exists and it is $r(S_{\alpha,\beta,\gamma})$.

From Remark 3.12, we know that $\{r(S_{\alpha_n,\beta_n,\gamma_n})\}_{n=1}^{+\infty}$ is contained in the compact set $[O, M]$, so it has an accumulation point y. Then there exists a subsequence

$$\{r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\}_{k=1}^{+\infty}$$

converging to y. Since $y \ge 0$ and $r(S_{\alpha,\beta,\gamma})$ is the only positive zero of $F_{\alpha,\beta,\gamma}$, we have

$$y = r(S_{\alpha,\beta,\gamma}) \Leftrightarrow \mathcal{F}(\alpha, \beta, \gamma, y) = F_{\alpha,\beta,\gamma}(y) = 0.$$

We can compute $\mathcal{F}(\alpha, \beta, \gamma, y)$:

$$\begin{gathered} \mathcal{F}(\alpha, \beta, \gamma, y) = \mathcal{F}\left(\alpha, \beta, \gamma, \lim_{k\to+\infty} r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\right) = \\ \mathcal{F}\left(\lim_{k\to+\infty}\left(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}, r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\right)\right) \overset{(\star)}{=} \\ \lim_{k\to+\infty} \mathcal{F}\left(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}, r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\right) = \\ \lim_{k\to+\infty} F_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}}\left(r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\right), \end{gathered}$$

where $(\star)$ follows from the continuity of $\mathcal{F}$. Now for every k

$$F_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}}\left(r(S_{\alpha_{n_k},\beta_{n_k},\gamma_{n_k}})\right) = 0,$$

so the limit is zero too and $y = r(S_{\alpha,\beta,\gamma})$.

Thus, every accumulation point of $\{r(S_{\alpha_n,\beta_n,\gamma_n})\}_{n=1}^{+\infty}$ is $r(S_{\alpha,\beta,\gamma})$, so the sequence converges and its limit is $r(S_{\alpha,\beta,\gamma})$. □

1.2. The minimum among triangular orbifolds. Up to now, we have been working with admissible cone-surfaces of signature $(0,3)$, without requiring the angles to be submultiples of 2π. In this section, we restrict ourselves to the orbifold case. We call *triangular orbifold* an orbifold of signature $(0,3)$; we want to find which one has smallest maximum injectivity radius.

NOTATION: we will write $(\alpha,\beta,\gamma) \leq (\alpha',\beta',\gamma')$ for $\alpha \leq \alpha', \beta \leq \beta'$ and $\gamma \leq \gamma'$.

LEMMA 3.14. *If $(\alpha,\beta,\gamma) \leq (\alpha',\beta',\gamma')$, then $r(S_{\alpha,\beta,\gamma}) \geq r(S_{\alpha',\beta',\gamma'})$ with equality if and only if $(\alpha,\beta,\gamma) = (\alpha',\beta',\gamma')$.*

PROOF. We pass from (α',β',γ') to (α,β,γ) in the following way

$$(\alpha',\beta',\gamma') \to (\alpha',\beta',\gamma) \to (\alpha',\beta,\gamma) \to (\alpha,\beta,\gamma).$$

and show that in any passage the radius decreases. Consider the first step. For all $x \geq 0$:

$$F_{\alpha',\beta',\gamma'}(x) - F_{\alpha',\beta',\gamma}(x) = 2(\cos\gamma' - \cos\gamma)\cosh^4 x \geq 0$$

because $\gamma' \leq \gamma$. So $F_{\alpha',\beta',\gamma'}(x) \geq F_{\alpha',\beta',\gamma}(x)$. Since $r(S_{\alpha',\beta',\gamma'})$ (resp. $r(S_{\alpha',\beta',\gamma})$) is the only solution in $(0,+\infty)$ of $F_{\alpha',\beta',\gamma'}$ (resp. $F_{\alpha',\beta',\gamma}$), we have

$$r(S_{\alpha',\beta',\gamma'}) \leq r(S_{\alpha',\beta',\gamma})$$

The same argument applied to all the steps shows that

$$r(S_{(\alpha,\beta,\gamma)}) \geq r(S_{(\alpha',\beta',\gamma')}).$$

□

LEMMA 3.15. *Given any triple of angles α,β,γ corresponding to a triangular surface $S_{\alpha,\beta,\gamma}$, (α,β,γ) is less or equal to (at least) one triple among $(\pi/2,\pi/3,\pi/7)$, $(\pi/2,\pi/4,\pi/5)$ and $(\pi/3,\pi/3,\pi/4)$.*

PROOF. Suppose $\alpha \geq \beta \geq \gamma$. By definition, all the angles are between $\pi/2$ and 0. Since $\alpha+\beta+\gamma < \pi$, either $\alpha = \pi/2$ and $\beta,\gamma < \pi/2$, or $\alpha < \pi/2$.

If $\alpha = \pi/2$, the condition $\alpha+\beta+\gamma < \pi$ implies that $\beta \leq \pi/3$. If $\beta = \pi/3$, then γ should be at most $\pi/7$, hence $(\alpha,\beta,\gamma) \leq (\pi/2,\pi/3,\pi/7)$. If $\beta < \pi/3$, i.e. $\beta \leq \pi/4$, then either $\beta = \pi/4$ and $\gamma \leq \pi/5$, or $\gamma \leq \beta \leq \pi/5$. In both situations, $(\alpha,\beta,\gamma) \leq (\pi/2,\pi/4,\pi/5)$.

If $\alpha < \pi/2$, then $\alpha,\beta,\gamma \leq \pi/3$. Moreover, since $\alpha+\beta+\gamma < \pi$ and γ is the smallest angle, then $\gamma \leq \pi/4$. So $(\alpha,\beta,\gamma) \leq (\pi/3,\pi/3,\pi/4)$, and this ends the proof. □

From Lemmas 3.14 and 3.15, it follows that it suffices to compare $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$, $S_{\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{5}}$ and $S_{\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{4}}$ to find the triangular orbifold with smallest maximum injectivity radius.

To prove that $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ is minimal, we first compute its maximum injectivity radius. We know that $\cosh r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)$ is the only solution (bigger than 1) of

$$2x\left(\cos\frac{\pi}{7}\,x^3-(2x^2-1)\sqrt{x^2-\frac{1}{4}}\right)=0,$$

i.e. the only solution bigger than 1 of

$$\cos\frac{\pi}{7}\,x^3=(2x^2-1)\sqrt{x^2-\frac{1}{4}}.$$

Both sides of the previous equation are positive for $x>1$, so an equivalent equation is

$$\left(\cos\frac{\pi}{7}\,x^3\right)^2=\left((2x^2-1)\sqrt{x^2-\frac{1}{4}}\right)^2.$$

We get a cubic equation in $t=x^2$:

$$\left(4-\cos^2\frac{\pi}{7}\right)t^3-5t^2+2t-\frac{1}{4}=0.$$

This equation has only one real solution t_0, so

$$r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)=\operatorname{arccosh}\sqrt{t_0}.$$

One can check that

$$F_{\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{5}}\left(r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)\right)<0 \text{ and } F_{\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{4}}\left(r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)\right)<0,$$

so

$$r\left(S_{\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{5}}\right)>r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right) \text{ and } r\left(S_{\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{4}})>r(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right).$$

We have proven the following:

PROPOSITION 3.16. *For every triangular orbifold S, $r(S)\geq r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)$, with equality if and only if $S=S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$.*

We define ρ_T to be $r\left(S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\right)$. Numerically

$$\rho_T\approx 0.187728\ldots$$

2. Orbifolds of signature $(g, n) \neq (0, 3)$

Even though we have only dealt with a small subset of the set of orbifolds, a good part of the work for this chapter is already done. We have a candidate minimal orbifold and we can even compute explicitly its maximum injectivity radius. Now we just need to show that this candidate is indeed minimal. To do this, it is enough to see that every other orbifold contains a disk of radius bigger than ρ_T.

Essentially, the idea is to find in each non-triangular orbifold some simple piece (a Y-piece or triangle) which is known to contain a disk of radius bigger than ρ_T. This works for most orbifolds, except for a couple of special cases which we have to treat separately.

2.1. Finding Y-pieces.

We use the following well known result:

PROPOSITION 3.17. *Every open Y-piece contains two closed disks of radius* $\rho_Y = \frac{\log 3}{2}$.

One can check that $F_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}(\rho_Y) < 0$, hence $\rho_Y > \rho_T$. As a consequence, if we find an embedded open Y-piece in a surface S, we know that $r(S) > \rho_T$.

PROPOSITION 3.18. *Let S be an orbifold of signature (g, n). Then S contains an embedded open Y-piece if and only if $g > 0$ and $3g + n \geq 5$ or $g = 0$ and $n \geq 6$.*

PROOF. [$\Rightarrow$] To have an embedded Y-piece, we need:

- if $g > 0$, at least two curves in a pants decomposition;
- if $g = 0$, at least three curves in a pants decomposition (since we cannot embed a one-holed torus).

Via Proposition 2.12, the number of curves in a pants decomposition is $3g - 3 + n$, so we get the stated conditions.

[$\Leftarrow$] Consider $S \setminus \Sigma$ as topological surface. We say that two pants decompositions $\mathcal{P}_1$ and $\mathcal{P}_2$ are joined by an elementary move (see [**HT80**]) if $\mathcal{P}_2$ can be obtained from $\mathcal{P}_1$ by removing a single curve and replacing it by a curve that intersects it minimally. Given a pants decomposition, we can construct its dual graph G (see [**Bus10**]): vertices correspond to pairs of pants and there is an edge between two vertices if the corresponding pairs of pants share a boundary curve. In particular, there is a loop at a vertex if two boundary curves of a pair of pants are glued to each other in the surface. The number of edges of G is the number of curves in a pants decomposition, i.e. $3g - 3 + n$, and every vertex in G has degree at most 3. Suppose $v \in G$ is a vertex of degree 3 and consider the associated pair of pants P. Passing

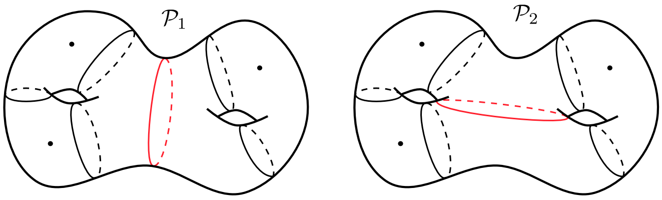


FIGURE 7. Two pants decompositions joined by an elementary move

to the geodesic representatives (via Theorem 2.11) of the boundary curves of P, we get a Y-piece, whose interior is embedded in S.

So, let us consider a surface S satisfying $g > 0$ and $3g + n \geq 5$ or $g = 0$ and $n \geq 6$, a pants decomposition on it and its dual graph.

If $g \geq 2$, the graph has at least two circuits (that could be loops). The circuits are connected to each other, since S is connected. So there is a vertex v on one of the circuits connected to a vertex outside of the circuit. Thus the degree of v is 3 and, as seen before, we get an embedded open Y-piece.

If $g = 1$, either the graph is a circuit with $3g - 3 + n \geq 2$ edges or it contains a circuit as a proper subgraph. In the first case, we can choose any edge and perform an elementary move on it to obtain a vertex of degree 3. In the second case, there is a vertex v on the circuit connected with a vertex outside the circuit, hence $\deg(v) = 3$. Given a vertex of degree 3 we have, as before, a Y-piece.

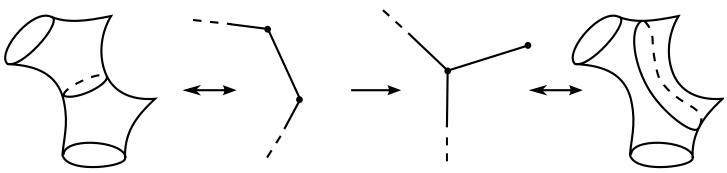


FIGURE 8. An elementary move producing a vertex of degree 3, for $g = 1$

If $g = 0$, the graph is a tree. If there is no vertex of degree three, the graph is a line with at least 3 edges. Again we can perform an elementary move on an edge between two vertices of degree 2 and get a vertex of degree 3, so a Y-piece in S. □

COROLLARY 3.19. *Every surface S of genus g with n singular points such that $g > 0$ and $3g + n \geq 5$ or $g = 0$ and $n \geq 6$ satisfies $r(S) > \rho_T$.*

2.2. Finding triangles in spheres with four or five singular points. Let T be a hyperbolic triangle and $r(T)$ the radius of its inscribed disk. It turns out that if the area of T is at least $\frac{\pi}{4}$, we have

$$F_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\left(\operatorname{arctanh}\left(\frac{1}{2}\sin\frac{\operatorname{area}(T)}{2}\right)\right) < 0.$$

By Proposition 2.6, this implies that $r(T) > \rho_T$.

Given a sphere S with n singular points, $n = 4$ or 5, we will look for an embedded triangle of area at least $\frac{\pi}{4}$. Let $\frac{2\pi}{p_1}, \dots \frac{2\pi}{p_n}$ be the angles at the singular points, where $p_i \geq 2$ is an integer or $p_i = \infty$ (i.e. the point is a cusp). Suppose $p_1 \leq p_2 \leq \cdots \leq p_n$. The area of the surface is

$$\operatorname{area}(S) = (n-2)2\pi - \sum_{i=1}^{n} \frac{2\pi}{p_i}.$$

If we cut the surface into $2(n-2)$ triangles (by cutting it first into two n-gons and then cutting each polygon into $n-2$ triangles), the average area of the triangles is

$$\frac{\operatorname{area}(S)}{2(n-2)} = \pi\left(1 - \frac{1}{n-2}\sum_{i=1}^{n}\frac{1}{p_i}\right).$$

This average is at least $\frac{\pi}{4}$ unless

(1) $n = 5$, $p_1 = p_2 = p_3 = p_4 = 2$ and $p_5 = 2$ or 3, or
(2) $n = 4$, $p_1 = p_2 = p_3 = 2$ and $p_4 < \infty$ or $p_1 = p_2 = 2$, $p_3 = 3$ and $p_4 \leq 5$.

So, if we are not in case (1) or (2), S contains a triangle with area at least $\frac{\pi}{4}$, hence $r(S) > \rho_T$.

In case (1), cut the surface along geodesics as in the following picture:

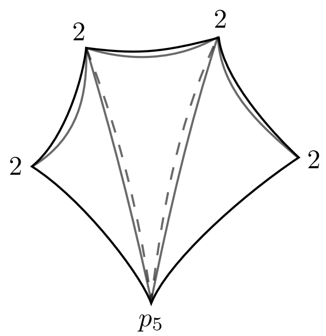


FIGURE 9. A way to cut a 5-punctured sphere into triangles

We obtain four triangles (if we cut open along a geodesic starting from a cone point of order two, we obtain an angle of π, so a side of a triangle).

The average area of those triangles is

$$\frac{\text{area}(S)}{4} = \frac{6\pi - \displaystyle\sum_{i=1}^{5} \frac{2\pi}{p_i}}{4} \geq \frac{\pi}{4},$$

so there is a triangle of area at least $\frac{\pi}{4}$, as desired.

In case (2), cut the surface along geodesics as in the following picture:

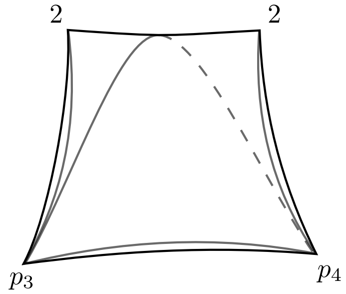


FIGURE 10. A way to cut a 4-punctured sphere into triangles

We have now two triangles and the average area is

$$\frac{\text{area}(S)}{2} = \frac{4\pi - \displaystyle\sum_{i=1}^{4} \frac{2\pi}{p_i}}{2} = \pi - \frac{\pi}{p_3} - \frac{\pi}{p_4}.$$

If $p_3 \geq 3$ or $p_4 \geq 4$, this average is at least $\frac{\pi}{4}$ and we have a disk of large enough radius. The only case that is left is $p_1 = p_2 = p_3 = 2$ and $p_4 = 3$, which we will consider separately.

2.3. Two special cases. *First special case:* let S be a sphere with three cone points of order 2 and one of order 3. Let $\tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ be $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ cut open along the side between the cone points of order 2 and 7. We want to embed $\tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ into S.

We decompose S into two isometric quadrilaterals by cutting along four geodesics between pairs of cone points. Each of the two quadrilaterals has three right angles and one angle $\frac{\pi}{3}$. Let λ, μ, l and m be the length of the sides as in picture 11. Using hyperbolic trigonometry, one can express λ and μ as functions of l and it is straightforward from the obtained expressions to deduce that $\lambda, \mu > \frac{\log 3}{2}$. Without loss of generality, assume $\lambda \geq \mu$. Since

$$\tanh \lambda \tanh \mu = \frac{1}{2},$$

we have

$$\lambda \geq \operatorname{arctanh} \frac{1}{\sqrt{2}}.$$

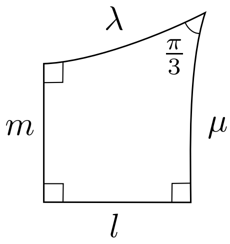


FIGURE 11. A trirectangle obtained by cutting the sphere S

It is possible to check that

$$\frac{\log 3}{2} > c$$
$$\operatorname{arctanh} \frac{1}{\sqrt{2}} > a$$

where a, b and c are the sides opposite to $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{7}$ in one of the triangles forming $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$. So $\mu > c$ and $\lambda > a$ and we can embed the triangle in the quadrilateral by placing B on the cone point of order three, c on the side of lenght μ and a on the side of length λ. By embedding in the same

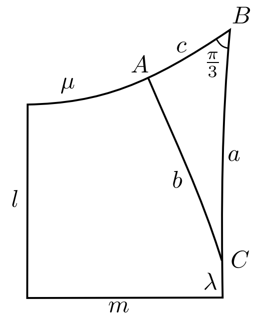


FIGURE 12. The embedding of the triangle in the quadrilateral

way another copy of the triangle in the other quadrilateral, we obtain the embedding of $\tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ in S.

Given a point on $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ that realizes the maximum injectivity radius, consider the corresponding point p on $\tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}} \subseteq S$. We know that $D_{\rho_T}(p) \cap \tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$ is embedded in $\tilde{S}_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$, hence in S. Note that the distance of p to the order two points is bigger than $d(p,m)$ or $d(p,l)$. So it's enough to prove that the distances $d(p,l)$ and $d(p,m)$ are strictly bigger than ρ_T to deduce that $D_{\rho_T}(p)$ is embedded in S.

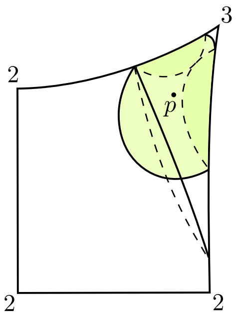


FIGURE 13. The disk in the surface S

We have:

$$\mu - c > \frac{\log 3}{2} - c$$

and $F_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\left(\frac{\log 3}{2} - c\right) < 0$, hence $\mu - c > \frac{\log 3}{2} - c > \rho_T$. Similarly

$$\lambda - a \geq \operatorname{arctanh} \frac{1}{\sqrt{2}} - a$$

and

$$F_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}\left(\operatorname{arctanh} \frac{1}{\sqrt{2})} - a\right) < 0,$$

so

$$\lambda - a \geq \operatorname{arctanh} \frac{1}{\sqrt{2}} - a > \rho_T.$$

Since $d(p, l) > \mu - c > \rho_T$ and $d(p, m) > \lambda - a > \rho_T$, $D_{\rho_T}(p)$ is embedded in S.

This shows that $r(S) \geq \rho_T$. Actually, since $d(p, l)$ and $d(p, m)$ are strictly bigger than ρ_T, we can choose a point p' in a small neighborhood of p such that

$$\begin{cases} d(p', l) > \rho_T \\ d(p', m) > \rho_T \\ d(p', B) > d(p, B), \end{cases}$$

where B is the cone point of order 3. Since we are increasing the distance from B, with the same argument used for triangular surfaces we get that $r_{p'} > r_p = \rho_T$. So $r(S) > \rho_T$.

Second special case: tori with exactly one singular point. We will use the following result [**Par06**]:

PROPOSITION 3.20. *Let P be a right-angled pentagon and $\mathring{P}$ its interior. Then $\mathring{P}$ contains a close disk of radius* $\rho_P = \frac{1}{2} \log \frac{9 + 4\sqrt{2}}{7}$.

Note that $\rho_P > \rho_T$, as $F_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}(\rho_P) < 0$. So again, the idea is to find a right-angled pentagon in any torus $\mathbb{T}$ with a singular point, to deduce that $r(\mathbb{T}) > \rho_T$. We proceed as follows.

First of all, we choose a simple closed geodesic on $\mathbb{T}$ and we cut open along it; we obtain a V-piece. Following [**Dia00**], this can be divided into two isometric pentagons with four right angles and an angle $\frac{\pi}{n}$ or 0.

LEMMA 3.21. *Let P be a pentagon with four right angles and an angle $\alpha \in [0, \frac{\pi}{2}]$. Then P contains a right-angled pentagon.*

PROOF. If $\alpha = \frac{\pi}{2}$, the result is trivial. Let us suppose that $\alpha < \frac{\pi}{2}$. Label the sides of the pentagon as in figure 14 and consider h, the common

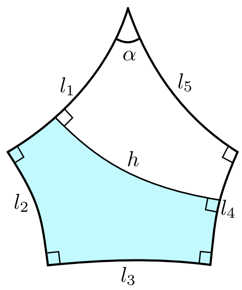


FIGURE 14. The pentagon $\mathcal{P}$ with the embedded right-angled pentagon

orthogonal of l_1 and l_4. The pentagon bounded by l_1, l_2, l_3, l_4 and h is a right-angled pentagon contained in P. □

With this result we can get the following.

LEMMA 3.22. *Let $\mathbb{T}$ be a torus with a singular point. Then $r(\mathbb{T}) > \rho_T$.*

PROOF. As we described before, we can find in $\mathbb{T}$ a pentagon with four right angles and an angle $\alpha \in [0, \frac{\pi}{2}]$. Via Lemma 3.21 this pentagon contains a right-angled pentagon, which contains a disk of radius

$$\rho_P > \rho_T$$

by Proposition 3.20. So

$$r(\mathbb{T}) \geq \rho_P > \rho_T.$$

□

3. Main results

We are now ready to state the main results of this chapter. The first one is a sharp lower bound for the maximum injectivity radius of 2-dimensional hyperbolic orbifolds.

THEOREM 3.23. *There exists a universal constant ρ_T such that every orbifold S satisfies*

$$r(S) \geq \rho_T$$

and the equality is attained if and only if S is a sphere with three cone points of orders 2, 3 and 7.

PROOF. If S has signature $(0,3)$, the result holds by Proposition 3.16. If $(g,n) \neq (0,3)$, we know that $r(S) > \rho_T$ by Corollary 3.19 and sections 2.2 and 2.3. □

Theorem 3.23 has a consequence about the action of the group of isometries of a surface on the surface itself, as we explain now.

Consider a hyperbolic surface S and its automorphism group $\mathrm{Aut}^+(S)$. Take a point $p \in S$ which is not fixed by any automorphism of S; then there exists $r > 0$ such that the disks $D_r(\varphi(p))$, where φ varies in $\mathrm{Aut}^+(S)$, are all embedded and pairwise disjoint, i.e. all images of p are at distance at least $2r$ from p. The question is: can we choose a radius which doesn't depend on the surface, provided we choose carefully the point p? And what is the maximum radius we can choose? Using Theorem 3.23, we can answer both questions.

Denote by $\rho(S)$ the maximum radius r for which there exists a point p such that $\{D_r(\varphi(p))\}_{\varphi \in \mathrm{Aut}^+(S)}$ are embedded and pairwise disjoint. Recall that a *Hurwitz surface* is a closed hyperbolic surface S such that

$$|\,\mathrm{Aut}^+(S)| = 84(g-1),$$

where g is the genus of S. Note that this means that the automorphism group is as large as possible, according to Hurwitz Theorem.

THEOREM 3.24. *For every hyperbolic surface S, we have $\rho(S) \geq \rho_T$, with equality if and only if S is a Hurwitz surface.*

PROOF. Given a surface S, we consider the quotient $S/\mathrm{Aut}^+(S)$ and the canonical projection $\pi : S \to S/\mathrm{Aut}^+(S)$. Note that for every $p \in S$, the disks $\{D_\rho(\varphi(p))\}_{\varphi \in \mathrm{Aut}^+(S)}$ are embedded and pairwise disjoint if and only if ρ is at most the injectivity radius of $\pi(p)$. To maximize the radius ρ, we choose $q \in S/\mathrm{Aut}^+(S)$ which has maximum injectivity radius and

$p \in \pi^{-1}(q)$. Moreover, S is a Hurwitz surface if and only if its quotient $S \big/ \mathrm{Aut}^+(S)$ is $S_{\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7}}$. Then the result follows from Theorem 3.23. $\square$

CHAPTER 4

Maximum injectivity radius of surfaces

In this chapter we will discuss the problem of bounding the maximum injectivity radius of hyperbolic surfaces. We will describe some known results and we will give a new proof of one of them.

1. Known results

The maximum injectivity radius of hyperbolic surfaces has been studied for a long time. One of the first results dates back to 1982, when Yamada proves in [**Yam82**] a sharp lower bound for all orientable hyperbolic surfaces.

THEOREM 4.1. *For any hyperbolic surface S, we have*

$$r(S) \geq \rho_S := \operatorname{arcsinh} \frac{2}{\sqrt{3}}$$

with equality if an only if S is a thrice-punctured sphere.

From this, it is easy to deduce that ρ_S is an infimum in each moduli space:

COROLLARY 4.2. *For every (g, n) such that $2 - 2g - n < 0$, we have*

$$\inf_{S \in \mathcal{M}_{g,n}} r(S) = \rho_S.$$

The infimum is not attained, unless the signature is $(0, 3)$.

PROOF. For $(g, n) = (0, 3)$, it is obvious from Theorem 4.1. Suppose $(g, n) \neq (0, 3)$; the idea is to pinch a pants decomposition on a surface to approach a union of thrice-punctured spheres. More precisely: by Theorem 4.1 we already know that $\inf_{S \in \mathcal{M}_{g,n}} r(S) \geq \rho_S$. So it is enough to construct a sequence of surfaces in $\mathcal{M}_{g,n}$ with maximum injectivity radius converging to ρ_Y. Fix any $S \in \mathcal{M}_{g,n}$ and a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3+n}\}$. Construct a sequence of surfaces $S_n \in \mathcal{M}_{g,n}$ such that

$$\ell_{S_n}(\gamma_i) = \frac{1}{n} \ell_S(\gamma_i)$$

and the twist parameters are the same as for S (see [**Bus10**] for the definition of twist parameters). For n going to infinity, S_n converges to a maximally noded surface at the boundary of moduli space. For n big enough, all curves in the pants decompositions are very short, so a maximal disk doesn't intersect $\mathcal{P}$, i.e. it is contained in a Y-piece with short boundary curves. Computations as in section 2 show that for the lengths of the boundary curves going to zero, the radius of a maximal disk embedded in a Y-piece tends to ρ_S. Thus $r(S_n)$ converges to ρ_S. □

Note that the situation is different for orbifolds, meaning that not for all signatures

$$\inf_{S\in\mathcal{O}_{g,n}} r(S) = \rho_T.$$

For instance, if $n = 0$, $\mathcal{O}_{g,0} = \mathcal{M}_{g,0}$, so the infimum is ρ_S by Theorem 4.1. In general, it is not known which is the infimum per signature and whether it is attained or not.

If we consider surfaces with boundary, we still have a lower bound on the maximum injectivity radius, which follows from a more general result of Parlier [**Par06**].

Theorem 4.3. *Let S be a closed hyperbolic surface of genus g and let γ be a simple closed geodesic on S. Then $S \setminus \gamma$ contains $4g-4$ closed disks of radius $\rho_Y := \frac{\log 3}{2}$. Conversely, if $\rho > \rho_Y$ is a given constant, there exists a simple closed geodesic γ_ρ on S such that $S \setminus \gamma_\rho$ does not contain any open disk of radius ρ.*

In terms of maximum injectivity radius, this implies that for every hyperbolic surface S with boundary, $r(S) \geq \rho_Y$.

If we are interested in upper bounds, the first result has been obtained in the case of closed (orientable and non-orientable) surfaces by Bavard in [**Bav96**], where he shows the following:

Theorem 4.4. *(a) Let S be a closed orientable hyperbolic surface of genus g. Then*

$$\cosh r(S) \leq \frac{1}{2\sin\frac{\pi}{12g-6}}$$

and for every genus g there exists an orientable surface of genus g realizing the equality.

(b) Let S be a closed non-orientable hyperbolic surface of Euler characteristic χ. Then

$$\cosh r(S) \leq \frac{1}{2\sin\frac{\pi}{6-6\chi}}$$

and for every Euler characteristic χ there exists a non-orientable surface of Euler characteristic χ realizing the equality.

More generally, there is an upper bound for all signatures, which has been proven by DeBlois in [**DeB15**].

THEOREM 4.5. *For $r > 0$, let*

$$\alpha(r) = 2\arcsin\frac{1}{2\cosh r}$$
$$\beta(r) = \arcsin\frac{1}{\cosh r}.$$

A hyperbolic surface S of signature (g, n) has maximum injectivity radius at most $r_{g,n}$, where $r_{g,n} > 0$ satisfies

$$3(4g + n - 2)\alpha(r_{g,n}) + 2n\beta(r_{g,n}) = 2\pi.$$

Moreover, for every (g, n) with $2g - 2 + n < 0$

$$0 \neq |\{S \in \mathcal{M}_{g,n} \,|\, r(S) = r_{g,n}\}| < \infty.$$

In particular, this means that

$$\sup_{S\in\mathcal{M}_{g,n}} r(S) = \max_{S\in\mathcal{M}_{g,n}} r(S) = r_{g,n}.$$

Note that the upper bound depends on the signature, while the lower bound of Theorem 4.1 doesn't.

2. A new proof of Yamada's Theorem

In this section we want to show how the techniques developed for proving Theorem 3.23 (in particular for the results of section 1) can be used to obtain Yamada's lower bound (Theorem 4.1). We get a new short proof, with simple geometric interpretation.

Note that recently Gendulphe [**Gen14**] has given another proof of Theorem 4.1.

PROOF OF THEOREM 4.1. Consider a hyperbolic surface S and fix a point p with maximum injectivity radius. Then the disk $D_{r_p}(S)$ determines at least two loops based at p, geodesics except at p, of length twice the injectivity radius. As for triangular surfaces, a careful analysis of these loops will give us the desired result.

Suppose there are exactly two such loops and assume that

- either at least one loop is homotopic to a cups, or
- the geodesic representatives α and β do not intersect.

Then we are in the situation of Figure 1, where α, β or γ can be cusps:

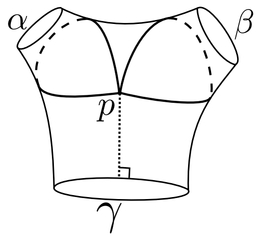


FIGURE 1. The point p with the two loops

Fix a point q in an ε-neighborhood of p which belongs to the orthogonal from p to γ, if γ is a simple closed geodesic, or to the geodesic from p to γ, if γ is a cusp. By choosing ε small enough, the loops based at q and surrounding α and β will be longer than the loops based at p, and by continuity all other loops based at q will still be longer than $2r_p$. So $r_q > r_p$, a contradiction.

As a consequence, we have two possibilities:

(a) there are at least three loops of length $2r_p$ such that either two are homotopic two cusps or the three geodesic representatives do not intersect;
(b) there are two loops of length $2r_p$ with geodesic representatives which intersect once.

Note that case (b) cannot happen if the surface is a thrice-punctured sphere.

Case (a)

Consider three loops of length $2r_p$; they determine a 3- or a 4-holed sphere.

Suppose first that there are three loops which determine a 3-holed sphere. Then, as for triangular surfaces, we can write equations for p. Denote by α, β and γ the three boundary curves or cusps, by $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ the angles of the three loops at p and by θ the angle of the (equilateral) triangle whose sides are the three loops (see Figure 2). We have:

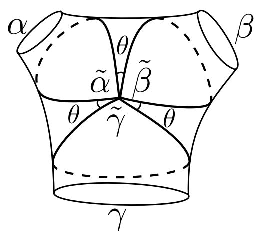


FIGURE 2. Three loops determining a Y-piece

$$
(4) \qquad \begin{cases}
\cosh \frac{\ell(\alpha)}{2} = \cosh r_p \sin \frac{\tilde{\alpha}}{2} \\
\cosh \frac{\ell(\beta)}{2} = \cosh r_p \sin \frac{\tilde{\beta}}{2} \\
\\
cosh \frac{\ell(\gamma)}{2} = \cosh r_p \sin \frac{\tilde{\gamma}}{2} \\
\cos \theta = 1 - \frac{1}{2\cosh^2 r_p} \\
\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + 3\theta = 2\pi
\end{cases}
$$

If by contradiction $r_p < \rho_S$, we get

$$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} > 2 \arcsin \sqrt{\frac{3}{7}}$$

and

$$\theta > \arccos \frac{11}{14},$$

so

$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + 3\theta > 2\pi$$

which is a contradiction. Moreover, $r_p = \rho_S$ is a solution if and only if $\ell(\alpha) = \ell(\beta) = \ell(\gamma) = 0$, i.e. if we are on a thrice-punctured sphere. So in this case, $r(S) \geq \rho_S$, with equality if and only if S is a thrice-punctured sphere.

If no three loops determine a 3-holed sphere, we choose any three loops and we denote by α, β and γ the corresponding geodesic representatives or cusps. Consider the associated 4-holed sphere and let δ be the fourth boundary curve or cusps. The loop based at p and homotopic to δ has length at least $2r_p$. We can write down equations satisfied by the pieces we obtain by cutting the four-holed sphere along the loops. If we assume by contradiction that $r_p \leq \rho_S$, we get (similarly to before)

$$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \geq 2 \arcsin \sqrt{\frac{3}{7}}.$$

Consider the quadrilateral with the four loops as sides; three sides have the same length $2r_p$ and the fourth has length at least $2r_p$. The two diagonals of the quadrilateral are longer than $2r_p$, otherwise we have three loops of length $2r_p$ determining a 3-holed sphere. Let us denote the angles as in Figure 3.

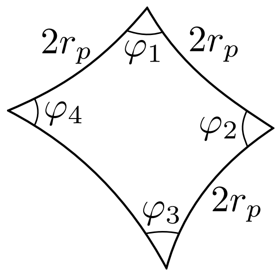


FIGURE 3. The quadrilateral with the four loops as sides

By hyperbolic trigonometry we get

$$\varphi_1, \varphi_2 > 2 \arcsin \frac{\sqrt{3}}{2\sqrt{7}}$$

and

$$\varphi_3, \varphi_4 > \arcsin \frac{\sqrt{3}}{2\sqrt{7}}.$$

So

$$\tilde{\alpha} + \tilde{b} + \tilde{\gamma} + \tilde{\delta} + 2\theta + 2\varphi > 2\pi,$$

a contradiction. Thus $r(S) > \rho_S$.

Case (b)

Assume again by contradiction that $r_p \leq \rho_S$. Consider the one-holed torus determined by the geodesic representatives α and β of the two loops of length $2r_p$. Denote its boundary curve or cusp by γ. Since $\alpha, \beta \leq 2\rho_S$, by the collar lemma (Theorem 2.9) we have $\alpha, \beta \geq 2 \operatorname{arcsinh} \frac{\sqrt{3}}{2}$.

Cut the one-holed torus along α and let d be the shortest path between the two copies of α. Again by Theorem 2.9, we obtain that

$$2 \operatorname{arcsinh} \frac{\sqrt{3}}{2} \leq \ell(d) \leq \ell(\beta) \leq 2\rho_S.$$

If γ is not a cusp, then

$$\cosh \frac{\ell(\gamma)}{2} = \sinh^2 \frac{\ell(\alpha)}{2} \cosh \ell(d) - \cosh^2 \frac{\ell(\alpha)}{2} =$$
$$= \sinh^2 \frac{\ell(\alpha)}{2} (\cosh \ell(d) - 1) + 1.$$

So $\frac{17}{8} \leq \cosh \ell(\gamma) \leq \frac{41}{9}$ and the width $w(\gamma)$ of the collar around γ satisfies

$$w(\gamma) > \log \frac{5}{4} > \log \frac{2}{\sqrt{3}}.$$

REMARK 4.6. *By hyperbolic trigonometry (Proposition 2.4), if a point p has distance at least $\log \frac{2}{\sqrt{3}}$ from the collar around a curve α of length less than $2\rho_S$, then the loop based at p and homotopic to α is at least $2\rho_S$.*

(i) If $\ell(\gamma) > 2\rho_S$, fix $q \in \gamma$. Consider a loop based at q of length $2r_q$ and its geodesic representative δ. There are three possibilities:
   - if $\delta \cap \gamma = \emptyset$, by Remark 4.6 we get $2r_q = \ell(\delta) > 2\rho_S$;
   - if $\delta \cap \gamma \neq \emptyset$ and $\delta \neq \gamma$, then δ crosses the one-holed torus, so it crosses α or β at least once and γ at least twice. Thus
   $$2r_q = \ell(\delta) \geq 2 \operatorname{arcsinh} \frac{\sqrt{3}}{2} + 4 \log \frac{5}{4} > 2\rho_S;$$
   - if $\delta = \gamma$, then $r_q > \rho_S$.

   In all cases, $r_q > \rho_S > r_p$, a contradiction.

(ii) If $\ell(\gamma) \leq 2\rho_S$, one can show that there exists a solution to the system (4), determining a point q with loops of length
$$\ell > 2 \operatorname{arccosh} \frac{\sqrt{37}}{3} > 2\rho_S.$$
Moreover
$$\cosh d(q, \alpha) = \frac{\sinh \frac{\ell}{2}}{\sinh \frac{\ell(\alpha)}{2)}} > \rho_S.$$
So there exists $r > \rho_S$ such that all loops based at q have length at least $2r$, thus again $r(S) > r_p$, a contradiction.

We are left with the case where γ is a cusp. In this situation, cut the one-holed torus along α and consider a point q which is

- equidistant from the two copies of α, and
- at distance $\log \frac{2}{\sqrt{3}} + \varepsilon$ from $\mathcal{H}_\gamma$ (where $\varepsilon > 0$ is small).

By explicit computations, $r_q > \rho_S > r_p$, which is again a contradiction.

So we have seen that in case (a), $r(S) \geq \rho_S$ with equality if and only if S is a thrice-punctured sphere. In case (b), S is not a thrice-punctured sphere and we always have $r(S) > \rho_S$. □

CHAPTER 5

Lengths of systoles

An very important object in the study of hyperbolic surfaces is the *systole*, i.e. a shortest essential closed curve. In this chapter and in the next one, we are going to study two main problems about the systole: bounds on its length and bounds on the number of systoles.

The first basic property of systoles is that, in most cases, they are simple closed geodesics.

PROPOSITION 5.1. *Let S be a hyperbolic surface of signature $(g, n) \neq (0, 3)$ and γ a systole. Then γ is simple.*

PROOF. By [**Bus10**, Theorem 4.2.4], if γ is not simple, then it is a figure-eight geodesic (i.e. a closed geodesic with exactly one self-intersection). Consider the three curves obtained as boundary of a small tubular neighborhood of γ. They cannot be contractible, otherwise we could shorten γ, and at least one of the three must be essential, otherwise S is a thrice-punctured sphere. Take the geodesic representative of this essential curve; it is shorter than γ, a contradiction. □

The only special case is the one of the thrice-punctured sphere, on which systoles are figure-eight geodesics of length $4 \operatorname{arcsinh} 1$. This follows from the fact that the thrice-punctured sphere does not contain any simple closed geodesic, from the fact that a shortest non-simple closed geodesic is a figure-eight geodesic ([**Bus10**, Theorem 4.2.4]) and from a result by Yamada [**Yam82**]:

PROPOSITION 5.2. *If a closed geodesic γ on a hyperbolic surface has self-intersection, then*

$$\ell(\gamma) \geq 4 \operatorname{arcsinh} 1$$

with equality if and only if S is the thrice-punctured sphere and γ is a figure-eight geodesic.

NOTE: to avoid this special case, from now on we will consider only surfaces surfaces of signature different from $(0, 3)$.

In the next two sections we will present some known results on the systole length and a new upper bound.

1. Known results

We denote by $\mathrm{sys}(S)$ the length of a systole of S. Note that it is easy to construct surfaces with arbitrarily short systoles in each moduli space. Indeed, we can set the lengths of all curves in a pants decomposition equal to ε. By the collar lemma, we are sure that these curves are actually the systoles for any chosen $\varepsilon < 2 \operatorname{arcsinh} 1$.

A much more challenging (and still open) problem is to find a sharp upper bound in terms of the signature. For closed surfaces, it is relatively easy to find an upper bound of order $2 \log g$ (see [**Bus10**, Lemma 5.2.1]).

LEMMA 5.3. *For any closed hyperbolic surface S of genus g, the systole length satisfies* $\mathrm{sys}(S) \leq 2 \log(4g - 2)$.

PROOF. Set $\ell = \mathrm{sys}(S)$. Any open disk of radius $\ell/2$ is embedded, thus

$$\mathrm{area}(D_{\ell/2}(p)) = 2\pi \left(\cosh \frac{\ell}{2} - 1 \right) \leq \mathrm{area}(S) = 4\pi(g-1),$$

which gives the desired bound. □

Even though this is a simple idea, this is essentially the best result that has been achieved so far for closed surfaces.

For surfaces with cusps, Schmutz Schaller ([**Sch94**]) has obtained the following upper bound:

THEOREM 5.4. *For $S \in \mathcal{M}_{g,n}$, with $n \geq 2$, we have*

$$\mathrm{sys}(S) \leq 4 \operatorname{arccosh} \frac{6g - 6 + 3n}{n}.$$

To get an idea of the growth of this bound, remember that for $x \to +\infty$

$$\operatorname{arccosh} x \sim \log x,$$

so for n fixed and g growing,

$$4 \operatorname{arccosh} \frac{6g - 6 + 3n}{n} \sim 4 \log g.$$

On the other hand, if g is fixed and n is growing, then the systole is bounded by a constant.

It is not easy to construct examples of surfaces with large systole. The first examples have been obtained by Buser and Sarnak in [**BS94**], where they show that there exist families of closed surfaces $\{S_k\}_k$ of genus g_k, with $g_k \to \infty$ as $k \to \infty$ such that

$$\mathrm{sys}(S_k) \geq \frac{4}{3} \log g_k.$$

There have been other constructions after Buser and Sarnak's one (see for instance Katz, Schaps and Vishne's paper [**KSV07**]), but the gap between $2\log g$ and $\frac{4}{3}\log g$ remains.

2. A new upper bound

The goal of this section is to prove that every surface of genus at least one has systole length bounded above by a function which only depends on the genus.

For any cusp c and any non-negative r, we denote by $\mathcal{H}_c$ the horoball of area two corresponding to c and we define the set $D_r(c)$ to be

$$D_r(c) := \{p \in S \mid d(p, \mathcal{H}_c) < r\} \cup \mathcal{H}_c.$$

If $D_r(c)$ is homeomorphic to a once-punctured disk, we can compute its area, which is

$$\text{area}(D_r(c)) = 2e^r.$$

LEMMA 5.5.
(a) If there are two cusps c and c' such that $D_r(c)$ and $D_r(c')$ are tangent, then the simple closed geodesic forming a pair of pants with them has length $4\operatorname{arccosh} e^r$, so

$$\text{sys}(S) \leq 4\operatorname{arccosh} e^r.$$

(b) If $D_r(c)$ is tangent to itself for some $r \geq \log 2$, then

$$\text{sys}(S) \leq 2\operatorname{arccosh}(e^r - 1).$$

PROOF. *(a)* Consider the pair of pants determined by the two cusps and the simple closed geodesic γ surrounding them. Cut it along the orthogonal from γ to itself, the shortest geodesic between the cusps and the perpendiculars from the cusps to γ. Consider one of the four obtained trirectangles; we denote its vertices by q, s, t and c and the intersection point of $\partial\mathcal{H}_c$ with a side by p, as in Figure 2.

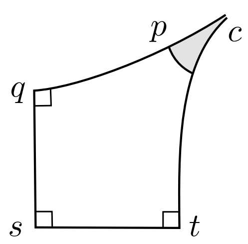


FIGURE 1. One of the trirectangles

Draw the quadrilateral in the upper half plane, choosing infinity as ideal point. We fix the two geodesics containing qc and tc to be $x = 0$ and $x = 1$. The area of $\mathcal{H}_c$ intersected with the quadrilateral is 1, so $\partial\mathcal{H}_c$ is given by $y = 1$ and $p = i$. Moreover, $d(p, q) = \frac{1}{2}d(\mathcal{H}_c, \mathcal{H}_c') = r$, so $q = ie^{-r}$. Consider $\mathcal{C}_1$ and $\mathcal{C}_2$, the Euclidean circles representing the geodesics through p and s and through s and t.

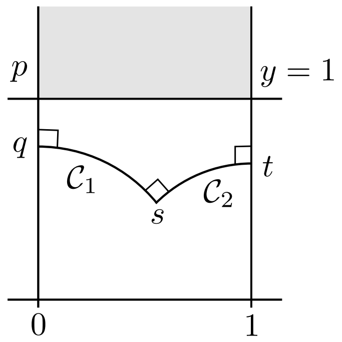


FIGURE 2. In the upper half plane

Since $\mathcal{C}_1 \perp \{x = 0\}$, $\mathcal{C}_2 \perp \{x = 1\}$ and $\mathcal{C}_1 \perp \mathcal{C}_2$, they have equations

$$\mathcal{C}_1 : x^2 + y^2 = R^2$$

and

$$\mathcal{C}_2 : (x - 1)^2 + y^2 = 1 - R^2$$

for some R. As $q \in \mathcal{C}_1$, we have $R = e^r$. By imposing $d(t, s) = \ell/4$, we obtain $\ell = 4 \operatorname{arccosh} e^r$.

(b) The cusp c with the curve of length $2r$ from $\mathcal{H}_c$ and back determines a pair of pants with at least one simple closed geodesic as boundary.

If the pair of pants has two cusps and a boundary curve α, we can cut it along the geodesic between the two cusps, the shortest geodesics between the cusps and α and the geodesic containing curve of length $2r$. We get two right-angled triangles with two ideal vertices and $\frac{\pi}{2}$ and two quadrilaterals with three right angles and an ideal vertex, as in Figure 3.

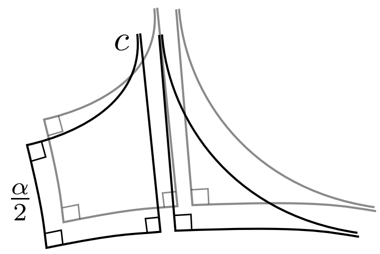


FIGURE 3. The cut pair of pants with two cusps

By direct computation similar to before, we obtain

$$\ell(\alpha) = 2 \operatorname{arccosh}(e^r - 1).$$

If the pair of pants has two boundary curves we denote them α and β, and we suppose that $\ell(\alpha) \leq \ell(\beta)$. We cut along the orthogonal from α to β, the shortest geodesics from α and β to the horoball and the geodesic containing the curve of length $2r$. We obtain four quadrilaterals, with three right angles and an ideal vertex, two by two isometric.

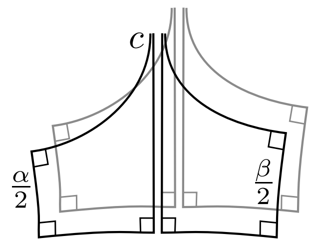


FIGURE 4. The cut pair of pants with one cusp

Again by direct computation we have

$$\ell(\alpha) = 2 \operatorname{arccosh}(ae^r)$$

and

$$\ell(\beta) = 2 \operatorname{arccosh}((1-a)e^r)$$

where a is the area of H_c intersected with one of the two quadrilaterals containing a part of α. Since $\ell(\alpha) \leq \ell(\beta)$, we have $a \leq \frac{1}{2}$. Moreover, α is longest when a is maximum, that is when $a = \frac{1}{2}$. In this case

$$\ell(\alpha) = \ell(\beta) = 2 \operatorname{arccosh}\left(\frac{e^r}{2}\right).$$

Since by assumption $r \geq \log 2$, we get that in both cases the curve α satisfies the $\ell(\alpha) \leq 2 \operatorname{arccosh}(e^r - 1)$. $\square$

REMARK 5.6. *From the proof of the lemma we also have that if $D_r(c)$ is tangent to itself for some $r \leq \log 2$, then* $\mathrm{sys}(S) \leq 2\,\mathrm{arcsinh}\, 1$.

We can now prove our bound on systole length for surfaces of genus $g \geq 1$.

THEOREM 5.7. *There exists a universal constant $K < 8$ such that every $S \in \mathcal{M}_{g,n}$ satisfies*

$$\mathrm{sys}(S) \leq 2\log g + K.$$

PROOF. Set $\ell = \mathrm{sys}(S)$.

For $n = 0$, Lemma 5.3 implies

$$\ell \leq 2\log g + 2\log 4.$$

Suppose now that $n \geq 1$. We split the proof into three non-mutually exclusive cases. The first situation we consider is when there are "many" cusps (how many will be made explicit): in this case two $D_c(r)$'s have to meet for a "small" r and will determine a short curve. In the second case, we assume that there are two cusps which are close to each other and the systole length will be bounded by the length of the curve surrounding them. In the final situation there are "few" cusps and we further assume any two are far away: in this case we show that there is a cusp with a short loop from its horoball to itself which in turn determines a short curve.

Case 1: $n \geq \sqrt{2\pi g}$

If the sets $D_r(c)$ are pairwise disjoint for different cusps c and each homeomorphic to a once-punctured disk, then

$$\mathrm{area}\left(\bigcup_{c \text{ cusp}} D_r(c)\right) = 2ne^r \leq \mathrm{area}(S) = 2\pi(2g+n-2)$$

thus

$$e^r \leq \frac{\pi(2g-2+n)}{n}.$$

Since $n \geq \sqrt{2\pi g}$, this implies

$$e^r \leq \frac{\sqrt{2\pi}(g-1)}{\sqrt{g}} + \pi.$$

So for some $r \leq \log\left(\frac{\sqrt{2\pi}(g-1)}{\sqrt{g}} + \pi\right)$ either two $D_r(c)$ are tangent to each other or one is tangent to itself. Lemma 5.5 now implies

$$\ell \leq 4\,\mathrm{arccosh}\left(\frac{\sqrt{2\pi}(g-1)}{\sqrt{g}} + \pi\right).$$

Case 2: there are two distinct cusps c_1 and c_2 with $d(\mathcal{H}_{c_1}, \mathcal{H}_{c_2}) \leq \log(2\pi(g-1+\sqrt{2\pi g}))$

By Lemma 5.5

$$\ell \leq 4 \operatorname{arccosh} \sqrt{2\pi(g-1+\sqrt{2\pi g})}$$

and we are done.

Case 3: $0 < n < \sqrt{2\pi g}$ and for any two cusps c_1, c_2 satisfy

$$d(\mathcal{H}_{c_1}, \mathcal{H}_{c_2}) > \log(2\pi(g-1+\sqrt{2\pi g}))$$

We fix a cusp c. Since any two cusps are far away, for $r \leq \log(2\pi(g-1+\sqrt{2\pi g}))$ the set$D_r(c)$ is disjoint from any other $\mathcal{H}_{c'}$. If it is also an embedded once-punctured disk, then

$$\text{area}(D_r(c)) = 2e^r \leq \text{area}(S) < 4\pi(g-1+\sqrt{2\pi g})$$

and so

$$e^r \leq \log(2\pi(g-1+\sqrt{2\pi g})).$$

We deduce that for some $r \leq \log(2\pi(g-1+\sqrt{2\pi g}))$, $D_r(c)$ is tangent to itself. By Remark 5.6, if $r \leq \log 2$ then $\ell \leq 2 \operatorname{arcsinh} 1$. Otherwise, by Lemma 5.5, we obtain

$$\ell \leq 2 \operatorname{arccosh}(2\pi(g-1+\sqrt{2\pi g})) - 1).$$

Now any surface with $n > 0$ will be in one of the three cases detailed above and as such we can deduce:

$$\ell \leq \max\left\{4 \operatorname{arccosh}\left(\sqrt{2\pi}(g-1)/\sqrt{g}+\pi\right), 4 \operatorname{arccosh} \sqrt{2\pi(g-1+\sqrt{2\pi g})}, \right.$$
$$\left. 2 \operatorname{arccosh}(2\pi(g-1+\sqrt{2\pi g})) - 1)\right\} < 2 \log g + 8.$$

□

Applying the techniques of the above theorem to punctured spheres, one can show that the systole length of punctured sphere is bounded by a uniform constant (which doesn't depend on the number of cusps). This is also a consequence of Theorem 5.4.

Note also that for $n \sim g^\alpha$, Schmutz Schaller's bound grows roughly like $4(1-\alpha)\log g$. So our bound is stronger for $\alpha < \frac{1}{2}$, while Schmutz Schaller's is better for $\alpha \geq \frac{1}{2}$.

CHAPTER 6

Kissing numbers

In the previous chapter we have addressed one of the first questions about systoles, i.e. how long they can be. In this chapter we consider another very natural question: how many systoles can a hyperbolic surface have?

The first to study the amount of systoles of surfaces was Schmutz Schaller, in analogy with classical lattice sphere packing problem in $\mathbb{R}^n$. In its simplest form, in $\mathbb{R}^2$, the problem is to bound the number of disjoint open disks of the same radius which are tangent to a given one (again of the same radius), such that all centers lie on some lattice Λ. The number of tangent disks is called the *kissing number* of the sphere packing (or of the lattice Λ). It is relatively easy to show that in this case the optimal bound is 6, which corresponds to disks with centers on the hexagonal lattice (i.e. the lattice generated by the vectors $(1,0)$ and $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$).

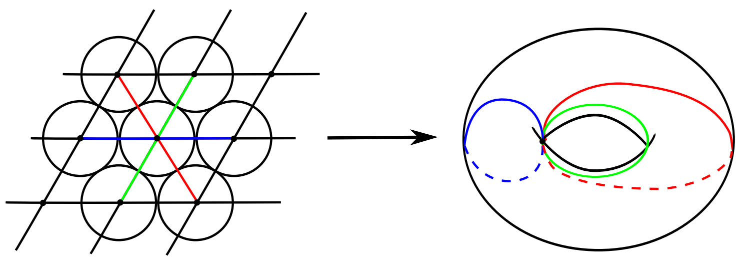


FIGURE 1. The hexagonal lattice in $\mathbb{R}^2$ and the corresponding quotient torus with three equal systoles

Note that each pair of tangent disks corresponds to a shortest non-zero vector of Λ. At the same time, each shortest vector also determines a shortest non-contractible curve on the quotient torus $\mathbb{R}^2/\Lambda$, i.e. a systole of the torus. As opposite vectors determine the same curve, we have that the kissing number of Λ is twice the number of homotopy classes of systoles of the associated flat torus. In particular, we know that a flat torus has at most three systoles, up to homotopy.

A possible generalization of this problem is to consider higher genus surfaces instead of tori. If we still require a constant curvature metric, we are bound to consider hyperbolic surfaces. As on hyperbolic surfaces there is at most one systole per homotopy class, the problem becomes counting the number of systoles on a hyperbolic surface.

Given a hyperbolic surface S, we denote by $\mathfrak{S}(S)$ the set of systoles of S. We define the *kissing number* $\mathrm{Kiss}(S)$ to be the number of systoles, i.e. the cardinality of $\mathfrak{S}(S)$.

As a first remark, note that surfaces with more than one systole (that is, with kissing number bigger than one) are rare. Indeed, McShane and Parlier proved in [**MP08**] that surfaces with two simple closed geodesics of the same length are a very small set in Teichmüller space: the set of surfaces with simple length spectrum is dense in Teichmüller space, and its complement is Baire meagre (i.e. a union of countably many nowhere dense sets).

On the other hand, it is easy to find surfaces with kissing number which grows linearly with g and n. In fact, take any surface of signature (g, n) where all curves of a pants decomposition are of some length $\varepsilon < 2 \operatorname{arcsinh} 1$ has kissing number $3g - 3 + n$: its systoles are precisely the curves in the pants decomposition (as at the beginning of section 1 of chapter 5).

Any linear lower bound on the kissing number is actually far from being optimal. Indeed, in [**SS97**], Schmutz Schaller proved that for every $\varepsilon > 0$ there is a family of closed surfaces S_k of genus g_k, with $g_k \to \infty$ for $k \to \infty$, such that

$$\mathrm{Kiss}(S_k) \geq g_k^{\frac{4}{3}-\varepsilon}.$$

Similar lower bounds can also be obtained for surfaces with cusps, by considering principal congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ (see [**Sch94**]). In these examples, the number of cusps grows roughly like $g^{\frac{2}{3}}$.

As for the length of the systole, there is a gap between the known lower and upper bounds. For closed surfaces, the best upper bound has been obtained by Parlier in [**Par13**]:

THEOREM 6.1. *There is a constant $C > 0$ such that any closed hyperbolic surface S of genus g has at most $C\frac{g^2}{\log g}$ systoles.*

This is actually a consequence of the upper bound on the systole of Proposition 5.3 and the following result in the same paper of Parlier.

THEOREM 6.2. *There is a constant $U > 0$ such that for any closed hyperbolic surface S of systole length* $\mathrm{sys}(S) = \ell$ *the following holds:*

$$\mathrm{Kiss}(S) \leq U \frac{e^{\ell/2}}{\ell} g.$$

As a consequence of Theorem 6.2, surfaces with "many" ($\sim g^{1+\alpha}$, $\alpha > 0$) systoles must have "long" ($\sim \log g$) systoles. Moreover, in [**Sch94**] and

[**Sch96**], Schmutz Schaller proved that surfaces with maximal systole length have many systoles (that is, at least $6g - 5 + 2n$). Furthermore, all known constructions of surfaces with long systoles or with large kissing number come from arithmetic constructions. All of this suggests a strong connection between the kissing number problem and the systole length problem.

Remark 6.3. *It is well known that systoles on a closed surface pairwise intersect at most once. So the kissing number problem is related to counting how many pairwise non-homotopic simple closed curves there can be on a (closed) surface, such that they pairwise intersect at most once.*

This problem, attributed to Farb and Leininger (see [**MRT14**]*), seems surprisingly difficult to solve. The best known upper bound is cubic in the genus and is proven by Przytycki in* [**Prz15**]*:*

Theorem 6.4. *The cardinality of a set $\mathcal{C}$ of essential simple closed curves on a surface S that are pairwise disjoint and intersecting at most once is at most*

$$g(4|\chi(S)|(|\chi(S)| + 1) + 1) + |\chi(S)| - 1.$$

On the other hand, Malestein, Rivin and Theran in [**MRT14**] *have constructed examples of such sets with size quadratic in the Euler characteristic. This result, together with the sub-quadratic bound in Theorem 6.1, shows that the purely topological problem is actually different then the kissing number one.*

Note that before this, it was already known that the topological and the systolic conditions were quite different: in fact, Anderson, Parlier and Pettet showed in [**APP11**] *that there are configurations of homotopy classes of curves that fail to be systoles for any hyperbolic metric on the surface.*

The objective of the remainder of this chapter is to prove an upper bound for the kissing numbers of all finite area surfaces. To do so, we start with a section about some intersection properties of systoles in this setting.

1. Intersection properties of systoles

We already mentioned that systoles on closed surfaces pairwise intersect at most once. On surfaces with cusps, this not necessarily the case. For instance, on punctured spheres, it is not difficult to see that systoles can intersect twice (the simplest case is a four times punctured sphere with at least two systoles – they necessarily intersect and the minimal intersection number between two distinct curves is 2). This phenomenon also occurs for surfaces with positive genus. An example of this can be derived from Buser's hairy torus (cf. [**Bus10**, Chapter 5]) with cusps instead of boundary curves and explicit examples in all genera are given in the sequel.

On the other hand, since systole length is bounded within each moduli space, it follows from the collar lemma that the intersection number between any two systoles is also bounded. This can be considerably sharpened: the first main result of the this section will be that two systoles on punctured surfaces can intersect at most twice.

Let α and β be simple closed geodesics on a surface S with $i(\alpha, \beta) \geq 2$ and fix orientations on them. The curve α divides β into arcs between consecutive intersection points. We say such an arc is of *type I* if the orientations at the two intersection points are different and of *type II* if the orientations are the same.

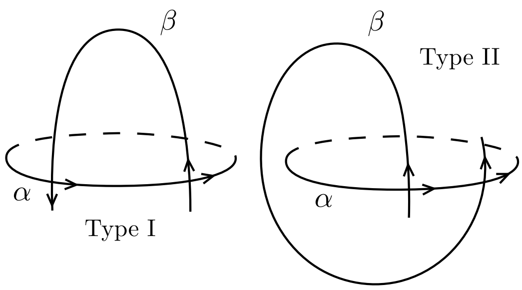


FIGURE 2. The two kinds of arcs

Note that the orientation at each intersection point depends on the choice of orientations of α and β, but being of type I or II is independent on the choice of orientations.

LEMMA 6.5. *If α and β are systoles of a surface $S \in \mathcal{M}_{g,n}$ with $i(\alpha, \beta) \geq 2$, all arcs between consecutive intersection points are of type I.*

PROOF. By contradiction, suppose that β contains arcs of type II. If there are at least two of them, there exists one, say β_1, of length at most $\frac{1}{2}\operatorname{sys}(S)$. Since β_1 divides α into two arcs, one of the two is of length at most $\frac{1}{2}\operatorname{sys}(S)$. Call this arc α_1 and consider the curve $\alpha_1 \cup \beta_1$.

If $\alpha_1 \cup \beta_1$ were essential, its geodesic representative would be shorter than $\operatorname{sys}(S)$, which is impossible. Thus $\alpha_1 \cup \beta_1$ must be non-essential. However one can construct a curve γ homotopic to $\alpha_1 \cup \beta_1$ such that $|\gamma \cap \alpha| = 1$, so via the bigon criterion γ and α intersect minimally. Thus

$$i(\gamma, \alpha) = 1$$

and as such γ is non-trivial in homology and is therefore essential, a contradiction.

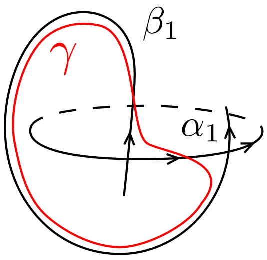


FIGURE 3. The arcs α_1 and β_1 with the curve γ

If there is exactly one arc β_1 of type II, there should be at least two (consecutive) arcs β_2 and β_3 of type I. Then if $\ell(\beta_1) \leq \frac{1}{2}\operatorname{sys}(S)$, we can argue as before to obtain a contradiction. If not, then $\ell(\beta_2 \cup \beta_3) \leq \frac{1}{2}\operatorname{sys}(S)$. The arcs β_2, β_3 and α determine an embedded four-holed sphere with a non-trivial curve of length at most $\frac{1}{2}\operatorname{sys}(S)$. By construction, the geodesic in the isotopy class of this curve is strictly shorter than the systole, a contradiction. □

PROPOSITION 6.6. *If α and β are systoles of $S \in \mathcal{M}_{g,n}$, then $i(\alpha, \beta) \leq 2$.*

PROOF. Suppose by contradiction that $i(\alpha, \beta) > 2$. By Lemma 6.5, all arcs between consecutive intersection points are of type I, so $i(\alpha, \beta)$ is even. Thus there are at least four intersection points and at least four arcs of β between consecutive intersection points. This implies that there is an intersection point and two arcs β_1 and β_2 departing from it with $\ell(\beta_1 \cup \beta_2) \leq \frac{1}{2}\operatorname{sys}(S)$. We argue as in the proof of Lemma 6.5: β_1, β_2 and α determine an embedded four-holed sphere with a non-trivial curve of length at most $\frac{1}{2}\operatorname{sys}(S)$. By construction, the geodesic in the isotopy class of this curve is strictly shorter than the systole, a contradiction. □

We can also prove that if two systoles intersect twice, there is a topological constraint on their configuration.

PROPOSITION 6.7. *If two systoles α and β intersect twice, one of the two bounds two cusps.*

PROOF. The two curves cut each other into arcs α_1, α_2 and β_1, β_2. Without loss of generality, we can assume $\ell(\alpha_1) \leq \ell(\beta_1) \leq \frac{1}{2}\operatorname{sys}(S)$. Consider $\gamma_1 = \alpha_1 \cup \beta_1$ and $\gamma_2 = \alpha_1 \cup \beta_2$. As γ_1 and γ_2 do not surround bigons, they cannot be trivial and as they can be represented by curves of length strictly less than $\operatorname{sys}(S)$, they must both bound a cusp. Hence β bounds two cusps. □

From Proposition 6.7 we can easily deduce that systoles on surfaces with at most one cusp intersect at most once, so in particular we find again the classical result for the case of closed surfaces. In the case of tori this can be improved to show that a surface with twice intersecting systoles has at least three cusps.

LEMMA 6.8. *If $S \in \mathcal{M}_{1,2}$ and α, β are systoles of S, then $i(\alpha, \beta) \leq 1$.*

PROOF. Suppose two systoles α and β intersect twice. Then $\mathrm{sys}(S) \geq 4 \operatorname{arcsinh} 1$ (see [**GS05**]) and by Proposition 6.7 one of the two curves bounds two cusps. Cut the surface along α and consider the one-holed torus component. The length of the shortest closed geodesic γ in the one-holed torus which doesn't intersect α satisfies (see [**Par14**])

$$\cosh \frac{\ell(\gamma)}{2} \leq \cosh \frac{\ell(\alpha)}{6} + \frac{1}{2}$$

and $\ell(\gamma) \geq \mathrm{sys}(S) = \ell(\alpha)$, so

$$\cosh \frac{\ell(\alpha)}{2} \leq \cosh \frac{\ell(\alpha)}{6} + \frac{1}{2}$$

which contradicts $\ell(\alpha) \geq 4 \operatorname{arcsinh} 1$. □

On the other hand, we can prove that for every genus there is a punctured surface with systoles intersecting twice. The constructions will involve gluing ideal hyperbolic triangles. Any such triangle has a unique maximal embedded disk tangent to all three sides. We say that two such triangles are glued *without shear* if their embedded disks are tangent.

LEMMA 6.9. *For every $g \geq 0$, there exists $n(g) \in \mathbb{N}$ and a surface $S \in \mathcal{M}_{g,n(g)}$ with two systoles intersecting twice.*

PROOF. For $g = 0$, we can set $n(0) = 4$, as mentioned at the beginning of section 1: any four times punctured sphere with at least two systoles will satisfy the requirement. To show the existence of such a surface, pick any $S \in \mathcal{M}_{0,4}$. If it has only one systole γ, start increasing the length of γ. As the systole length of a punctured spheres is bounded above by a constant, if $\ell(\gamma)$ increases enough, γ is not a systole anymore, i.e. there is a simple closed geodesic which is shorter than γ. So there exists a value of $\ell(\gamma)$ such that γ is a systole and there exists a simple closed curve δ of the same length.

For $g \geq 1$, we use a building block constructed as follows. Consider a square and a triangulation of it with 32 triangles, given by first subdividing the square into a grid of 16 squares and then adding one diagonal for all squares as in Figure 4. Each of the triangles in the square will be replaced by an ideal hyperbolic triangle and all gluings will be without shear.

For $g = 1$, glue opposite sides of the square (again triangles are glued without shear) to obtain a torus with $n(1) = 16$ cusps.

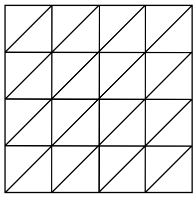


FIGURE 4. The triangulation of the square

For $g \geq 2$, consider a polygon obtained by gluing a $1 \times (g-1)$ rectangle and a $1 \times 2(g-1)$ rectangle along the long sides, as in Figure 5.

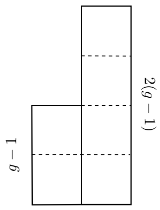


FIGURE 5. The polygon for $g = 3$

Think of this polygon as a $4g$-gon (with sides corresponding to sides of the squares). Fix an orientation and choose a starting side, to identify the $4g$ sides following the standard pattern $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ to obtain a genus g surface. If we now replace each 1×1 square by the building block (always gluing adjacent triangles without shear), we get a surface of genus g with a decomposition into $32 \cdot 3(g-1)$ ideal triangles. Since it is a triangulation, the number of edges is $\frac{3}{2} \cdot 32 \cdot 3(g-1)$. By an Euler characteristic argument, this implies that the surface has $n(g) = 46g - 46$ cusps.

For any $g \geq 1$, consider the set $\mathcal{C}_g$ of curves surrounding pairs of cusps which are connected by an edge between vertices of degree 6 in the triangulation of the surface. By construction, each of these intersects another such curve twice and we defer the proof that these curves are systoles to Lemma 6.10.

□

We now prove our claim that the curves in $\mathcal{C}_g$ are indeed systoles.

LEMMA 6.10. *For all $g \geq 1$, the curves in $\mathcal{C}_g$ are systoles.*

PROOF. Consider the triangulation of the surface. For $g = 1$, all vertices are of degree 6. When $g \geq 2$, the pasting scheme associates all exterior vertices of the $4g$-gon and the point in the quotient has degree $12g - 6$ (as we can check by applying the hand-shaking lemma to the graph given by the triangulation). The remaining vertices are all of degree 6. We denote by Γ the graph dual to the triangulation. From what we've just said, for $g = 1$, cutting the surface along Γ decomposes the surface into hexagons. When $g \geq 2$, cutting along Γ decomposes the surface into hexagons and a single $(12g - 6)$-gon.

Any simple closed oriented geodesic γ on the surface can be homotoped to a curve on Γ. At every vertex crossed by the curve, the orientations on the surface and on the curve give us a notion of "going left" or "going right". We can associate to γ a word w in the matrices $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where each L corresponds to a left turn and each R to a right turn. This way of understanding curves on "zero shear surfaces" is fully explained in [**BM04**]. In particular, Brooks and Makover show how to compute the length of these curves in terms of the associated word:

$$\ell(\gamma) = 2 \operatorname{arccosh} \left(\frac{\operatorname{Tr}(w)}{2} \right).$$

Note that if w is a word associated to a non-oriented curve γ, then also the word w', obtained by reading w backwards and replacing each L with an R and each R with an L, or any cyclic permutation of w and w' are associated to γ.

Each curve in $\mathcal{C}_g$ corresponds to the word $w_0 = RL^4RL^4$ (or LR^4LR^4, or any cyclic permutation of these, depending on the choice of an orientation and of a starting point on the curve), which via a simple computation has trace 34. To show that the curves in $\mathcal{C}_g$ are systoles, it is enough to show that all other words corresponding to simple closed geodesics have trace at least 34.

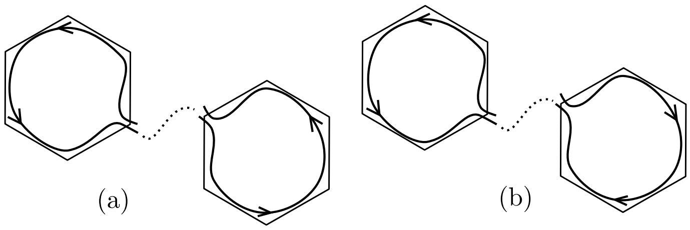
We use the following remark (see for instance [**Pet13**]):

Remark 6.11. *If a word can be written as* $w = \ldots w_1 \ldots w_2 \ldots w_k \ldots$, *then*

$$Tr(w) \geq Tr\left(w_{\sigma(1)} \ldots w_{\sigma(k)}\right)$$

for any cyclic permutation σ *of* $1, \ldots, k$.

Let γ be a simple closed geodesic which is not in $\mathcal{C}_g$. First we observe that we only need to consider curves represented by circuits in Γ. Indeed, if γ corresponds to a closed path which contains an essential (i.e. not corresponding to a curve going around a cusp) circuit γ', the word of γ will contain the word of γ'. By Remark 6.11, γ' is at most as long as γ and we can consider γ' instead. Otherwise, if γ is formed from non-essential circuits, it should contain at least two of them. Note that since non-essential circuits surround a cusp, they trace a hexagon or a $(12g-6)$-gon. If both these circuits surround hexagons, we are in one of the following situations:



In case (a), a word associated to the curve contains $RL^5 \dots RL^5$ and in case (b) it contains $LR^5 \dots RL^5$. In both cases, by Remark 6.11 and a simple computation, their traces are bigger than 34. Now if one of the two circuits surrounds the vertex of the triangulation of degree $12g-6$, the curve is even longer.

Suppose then that γ is represented by an essential circuit. If it passes through five consecutive edges of a hexagon (said differently, a corresponding word contains R^4) and is not in $\mathcal{C}_g$, the following modification of the curve (see figure 6) provides an essential circuit.

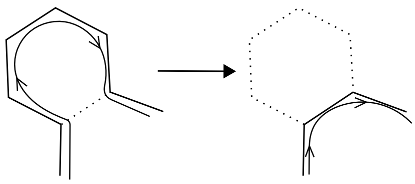


FIGURE 6. Shortening a curve

A word of the curve on the left contains LR^4L, while the one of the curve on the right contains R^2, so the trace decreases (again by Remark 6.11) and we obtain a shorter curve.

We now assume a word w representing γ does not contain L^4 or R^4 and as such it is made of blocks of type L^iR^j, for $1 \leq i,j \leq 3$. If w is made of four or more such blocks, then

$$\mathrm{Tr}(w) \geq \mathrm{Tr}((LR)^4) > 34.$$

Moreover, the length of w is at least 7, as the shortest circuits in Γ are of length 6 and correspond to curves surrounding cusps. With this in hand, one needs to check the finite set of words w made of blocks as above, of length at least 7, and of trace at most 33. To do this one can proceed as follows.

Consider first the case of words made by two blocks. The conditions above give the following system of equations:

$$\begin{cases} \mathrm{Tr}(L^i R^j L^k R^l) \leq 33 \\ 0 < i, j, k, l < 4 \\ i + j + k + l \geq 7 \end{cases}$$

where we only look for integer solutions. Note that as the cyclic permutation of a word represents the same curve as the word itself, we can add the condition $i + j \leq k + l$ to reduce the number of solutions. Now we can solve the system and obtain a set of words. We further reduce the number of works by remarking that $L^i R^j L^k R^l$ and $L^l R^k L^j R^i$ represent the same curve, just with a different choice of orientation. It is then straightforward to check that the curves corresponding to these words do not correspond to simple closed geodesics on the surface.

The case of words made by three blocks can be treated in the same way. □

2. Kissing number bound

The goal of this section is to prove an upper bound for the kissing number depending on the systole length and then deduce a bound depending only on the signature, using the known systole bounds.

As we have seen, systoles on punctured surfaces can have different topological configurations. This suggests a subdivision of the set of systoles into three sets and we will give separate bounds for the cardinalities of each subset.

We say that two simple closed geodesics α and β *bound a cusp* if they form a pair of pants with a cusp. We define:

$$A(S) := \{\alpha \in \mathfrak{S}(S) \mid \alpha \text{ bounds two cusps}\}$$
$$B(S) := \{\alpha \in \mathfrak{S}(S) \setminus A(S) \mid \exists \beta \in \mathfrak{S}(S) \setminus A(S) \text{ s.t. } \alpha \text{ and } \beta \text{ bound a cusp}\}$$
$$C(S) := \mathfrak{S}(S) \setminus (A(S) \cup B(S)).$$

Note that by Proposition 6.7 two systoles in $\mathfrak{S}(S) \setminus A(S)$ intersect at most once.

2.1. Bound on $|A(S)|$.

We have already seen in Lemma 5.5, a curve of length ℓ bounds two cusps c_1 and c_2 if and only if

$$d(\mathcal{H}_{c_1}, \mathcal{H}_{c_2}) = d(\ell) := 2 \log \cosh \frac{\ell}{4}.$$

To bound $|A(S)|$, we will bound the number of pairs of cusps at distance $d(\mathrm{sys}(S))$.

LEMMA 6.12. *Let S be a surface with $\mathrm{sys}(S) = \ell$ and c a cusp of S. There are at most $\lfloor 2\cosh(\ell/4)\rfloor$ cusps c' which satisfy $d(\mathcal{H}_c, \mathcal{H}_{c'}) = d(\ell)$.*

PROOF. Suppose c_1 and c_2 are two cusps such that

$$d(\mathcal{H}_c, \mathcal{H}_{c_1}) = d(\mathcal{H}_c, \mathcal{H}_{c_2}) = d(\ell).$$

Since $\mathrm{sys}(S) = \ell$, the distance between $\mathcal{H}_{c_1}$ and $\mathcal{H}_{c_2}$ is at least $d(\ell)$. Consider the segment α realizing the distance between $\mathcal{H}_c$ and $\mathcal{H}_{c_1}$, the segment β realizing the distance between $\mathcal{H}_c$ and $\mathcal{H}_{c_2}$ and the shortest arc γ of $\partial\mathcal{H}_c$ bounded by the endpoints of α and β.

Let δ be the unique geodesic segment freely homotopic with endpoints on $\partial\mathcal{H}_{c_1}$ and $\partial\mathcal{H}_{c_2}$ to the curve $\alpha \cup \beta \cup \gamma$. Then its length is at least $d(\ell)$.

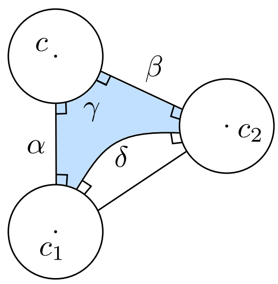


FIGURE 7. The non-geodesic hexagon

By a direct computation on the (non-geodesic) hexagon determined by α, β, δ and the three horocycles, one can show that

$$\ell(b) \geq \frac{1}{\cosh\frac{\ell}{4}}.$$

Since $\partial\mathcal{H}_c$ has length 2, the number of cusps around p at distance $d(\ell)$ is bounded above by

$$2 \Big/ \frac{1}{\cosh\frac{\ell}{4}},$$

which proves the claim as we are bounding an integer. $\square$

As a consequence, we get the following.

PROPOSITION 6.13. *For $S \in \mathcal{M}_{g,n}$ with $\mathrm{sys}(S) = \ell$*

$$|A(S)| \leq \frac{n}{2}\lfloor 2\cosh(\ell/4)\rfloor.$$

PROOF. There are n cusps, each of which can be surrounded by at most $\lfloor 2\cosh(\ell/4)\rfloor$ cusps at distance $d(\ell)$. The result follows as each curve surrounds two cusps. □

It is actually possible to get an upper bound for $|A(S)|$ in a simpler way.

LEMMA 6.14. *If S has signature (g,n), then $|A(S)| \leq 3(2g-2+n)$.*

PROOF. Consider the set of cusps; if there is a systole bounding them, we join them with a simple geodesic lying in the pair of pants determined by the systoles. We complete this set of geodesics into an ideal triangulation (i.e. a decomposition in ideal triangles) of the surface. The number of vertices of the triangulation is n, and if e is the number of edges, $e \geq |A(S)|$. Since we have a triangulation, the number of triangles is $\frac{2e}{3}$. The Euler characteristic of the compactified surface is $2-2g$, so

$$n - e + \frac{2e}{3} = 2 - 2g.$$

Thus

$$|A(S)| \leq e \leq 3(2g-2+n).$$

□

Interestingly, the bound of Lemma 6.14 can be deduced by the one of Proposition 6.13, plugging in the systole bound of Theorem 5.4. For surfaces of genus at least one, we will use the bound of Lemma 6.14, but we will need the sharper bound of Proposition 6.13 for punctured spheres.

2.2. Bound on $|B(S)|$. Fix a cusp c and define

$$B(c) := \{\alpha \in B(S) \mid \exists \beta \in B(S) \text{ s.t. } \alpha \text{ and } \beta \text{ bound } c\}.$$

Any curve $\alpha \in B(c)$ is at a fixed distance $D(\ell)$ from $\mathcal{H}_c$. By direct computation in the pair of pants bounded by α, c and some other $\beta \in B(c)$, one obtains

$$D(\ell) := \log \frac{2\cosh\frac{\ell}{2}}{\sinh\frac{\ell}{2}}.$$

Suppose that (α,β) and (γ,δ) are two pairs of systoles in $B(S)$ which bound c. Then γ has to pass through the pair of pants determined by α, β and c, so it must intersect α or β. Since curves in $\mathfrak{S}(S) \setminus A(S)$ pairwise intersect at most once, then $i(\alpha,\gamma) = i(\beta,\gamma) = 1$, and the same holds for δ.

LEMMA 6.15. *Let S be a surface of signature $(g,n) \neq (1,1)$. If α and β are systoles of length ℓ intersecting once, their angle of intersection satisfies*

$$\sin\angle(\alpha,\beta) \geq \sin\theta_\ell := \begin{cases} \frac{2}{\sqrt{5}}, & \ell < 2\operatorname{arccosh}\frac{3}{2} \\ \frac{\sqrt{2\cosh\frac{\ell}{2}+1}}{\cosh\frac{\ell}{2}+1}, & \ell \geq 2\operatorname{arccosh}\frac{3}{2}. \end{cases}$$

In particular, the angle of intersection is bounded below by a function θ_ℓ that behaves like $e^{-\ell/4}$ as ℓ goes to infinity.

Note that also [**Par06**, Lemma 2.4] gives a lower bound on the angle of intersection of two systoles intersecting once, with the same order of growth.

Proof. Consider the two systoles and the one holed torus T they determine. Since $(g,n) \neq (1,1)$, the boundary component δ of T is a simple closed geodesic.

As α and β are systoles of S, they are also systoles of T. As such they satisfy the systole bound for T which depends on the length of δ, namely (see [**Par14**])

$$\cosh \frac{\ell(\delta)}{6} \geq \cosh \frac{\ell}{2} - \frac{1}{2}.$$

We first consider the case when $\ell \geq 2 \operatorname{arccosh} \frac{3}{2}$. We have $\cosh \frac{\ell}{2} - \frac{1}{2} \geq 1$ and the condition stated above is non empty. Cut T along α and consider the shortest curve h connecting the two copies of α.

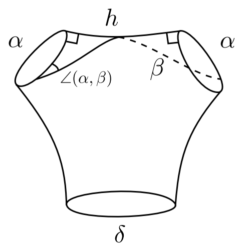


Figure 8. The torus T with the curves α, β and h

By hyperbolic trigonometry, using $\cosh \frac{\ell(\delta)}{6} \geq \cosh \frac{\ell}{2} - \frac{1}{2}$, a direct computation provides

$$\cosh h \geq \frac{4\cosh^2 \frac{\ell}{2} - \cosh \frac{\ell}{2} - 1}{\cosh \frac{\ell}{2} + 1}.$$

Now consider one of the two right-angled triangles determined by arcs of α, β and h. We have

$$\frac{\sinh \frac{h}{2}}{\sin \angle(\alpha,\beta)} = \sinh \frac{\ell}{2}$$

which, together with the estimate on h, yields

$$\sin \angle(\alpha,\beta) \geq \frac{\sqrt{2\cosh \frac{\ell}{2} + 1}}{\cosh \frac{\ell}{2} + 1}.$$

If $\ell < 2\,\mathrm{arccosh}\,\frac{3}{2}$, we deduce the inequality

$$\sin \angle(\alpha,\beta) \geq \frac{2}{\sqrt{5}}$$

by arguing as above, but replacing the estimate $\cosh \frac{\ell(\delta)}{6} \geq \cosh \frac{\ell}{2} - \frac{1}{2}$ by $\ell(\delta) \geq \ell$. $\square$

Fix now two systoles α and β which bound a cusp c; denote by $\mathcal{P}$ the pair of pants we obtain. As the two boundary curves of $\mathcal{P}$ have the same length, there is an isometric involution φ of $\mathcal{P}$ that sends α to β (the rotation of angle π around the cusp). If c is bounded by two other systoles γ and δ, the involution sends $\gamma \cap \mathcal{P}$ to $\delta \cap \mathcal{P}$, because of the symmetry of the pair of pants determined by γ, δ and c. If we quotient $\mathcal{P}$ by φ and we consider the image of $B(c)$, we get a set of geodesics at distance $D(\mathrm{sys}(S))$ from a horoball of area one, all pairwise intersecting with angle at least $\theta_{\mathrm{sys}(S)}$. This observation is crucial to show the following result.

Lemma 6.16. *If S is a surface of signature $(g,n) \neq (1,1)$ and systole length* $\mathrm{sys}(S) = \ell$, *for any cusp c the number of elements in $B(c)$ is bounded above by*

$$m(\ell) := \frac{\cosh \frac{\ell}{2}}{\sinh \frac{\ell}{2}} \frac{2}{\sin \frac{\theta_\ell}{2}}.$$

Proof. The situation is as in the following figure which locally represents the elements of $B(c)$ under the quotient by φ. Note that every element in the quotient by φ represents two elements from $B(c)$.

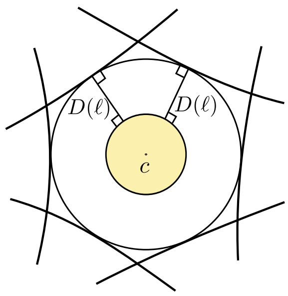


Figure 9. Geodesics around a horoball

The inner circle (which we will refer to as the inner horocycle) represents the quotient horoball of area one and the external one is the horocycle at distance $D(\ell)$ from the horoball of area one. By looking at the unique orthogonal geodesics between elements of $B(c)/\varphi$ and the inner horocyle, we can determine a cyclic ordering on the elements of $B(c)/\varphi$. Two neighboring geodesics with respect to this ordering, determine a *subarc* on the inner horocycle as follows. We consider the orthogonal geodesic between them and the inner horocycle and take the subarc of the horocyle which forms a pentagon with the two geodesics and the orthogonal (see Figure 2.2). By a direct computation, using the lower bound on the angle of intersection, this subarc on the inner horocycle is of length at least

$$\frac{\sinh \frac{\ell}{2}}{\cosh \frac{\ell}{2}} \sin \frac{\theta_\ell}{2}.$$

These subarcs are all disjoint and are in the same number as the elements of $B(c)\big/_{\varphi}$ (keep in mind that *any* two elements of $B(c)\big/_{\varphi}$ intersect).

From this we deduce an upper bound on $\left|B(c)\big/_{\varphi}\right|$:

$$\frac{1}{\frac{\sinh \frac{\ell}{2}}{\cosh \frac{\ell}{2}} \sin \frac{\theta_\ell}{2}}.$$

Now $2\left|B(c)\big/_{\varphi}\right| = |B(c)|$, which completes the proof. □

As a consequence, we obtain an upper bound on $|B(S)|$.

PROPOSITION 6.17. *If $S \in \mathcal{M}_{g,n}$, $(g,n) \neq (1,1)$, has systole of length $\mathrm{sys}(S) = \ell$, then*

$$|B(S)| \leq n\, m(\ell).$$

PROOF. We have

$$B(S) = \bigcup_{c \text{ cusp}} B(c)$$

and for every cusp c

$$|B(c)| \leq m(\ell).$$

□

2.3. Bound for $|C(S)|$. By definition, elements of $C(S)$ are systoles such that

- they pairwise intersect at most once and
- no two disjoint curves bound a cusp.

We follow a similar argument to one found in [**Par13**] to obtain an upper bound on $|C(S)|$. In particular we will need a collar lemma for systoles.

LEMMA 6.18. *Let $\mathrm{sys}(S) = \ell$ and consider α, $\beta \in C(S)$. If α and β do not intersect, then they are at distance at least $2r(\ell)$, where*

$$r(\ell) = \operatorname{arcsinh} \frac{1}{2 \sinh \frac{\ell}{4}}.$$

Proof. Fix a pair of pants with α and β as boundary and consider the third boundary component γ. Since α and β are in $C(S)$, they do not bound a cusp, so γ is a simple closed geodesic of length at least ℓ. The result follows by a standard trigonometric computation. $\square$

As a consequence, if α and β in $C(S)$ pass through the same disk of radius $r(\ell)$ then they intersect.

Moreover, we have seen in Lemma 6.15 that there is a lower bound on the angle of intersection of systoles intersecting once. With this in hand we prove the following.

Lemma 6.19. *If $(g,n) \neq (1,1)$, $\operatorname{sys}(S) = \ell$ and α and β in $C(S)$ pass through a disk of center p and radius $r(\ell)$, the distance between p and the point q of intersection between α and β satisfies*

$$d(p,q) \leq R(\ell),$$

where

$$\sinh R(\ell) = \begin{cases} \frac{5}{8\sinh\frac{\ell}{4}}, & \ell < 2\operatorname{arccosh}\frac{3}{2} \\ \frac{\cosh\frac{\ell}{2}+1}{2\sinh\frac{\ell}{4}\sqrt{2\cosh\frac{\ell}{2}+1}}, & \ell \geq 2\operatorname{arccosh}\frac{3}{2}. \end{cases}$$

Note that $R(\ell)$ is bounded for $\ell \geq 2\operatorname{arcsinh} 1$.

Proof. The proof is analogous to the proof of [**Par06**, Lemma 2.6]. Fix $p_\alpha \in \alpha$ and $p_\beta \in \beta$ lying in $D_{r(\ell)}(p)$. We have two triangles of vertices p, p_α, q and p, p_β, q, and the sum of the two angles θ_α and θ_β at q is the angle of intersection $\angle(\alpha,\beta)$. Suppose $\theta_\alpha \geq \frac{\angle(\alpha,\beta)}{2}$ and consider the angle η of the triangle p, p_α, q at p_α.

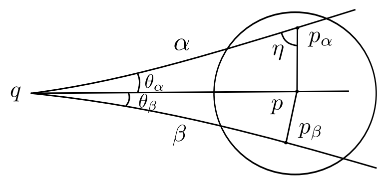


Figure 10. α and β passing though a disk of radius $r(\ell)$

Then

$$\frac{\sin\eta}{\sinh d(p,q)} = \frac{\sin\theta_\alpha}{\sinh d(p,p_\alpha)}.$$

Using $\theta_\alpha \geq \frac{\angle(\alpha,\beta)}{2}$, $d(p,p_\alpha) < r(\ell)$ and Lemma 6.15, we obtain the claimed result. $\square$

We are now in a position to obtain a bound on $|C(S)|$.

PROPOSITION 6.20. *If* $S \in \mathcal{M}_{g,n}$*,* $g \neq 0$ *and* $(g,n) \neq (1,1)$*, has systole of length* $\mathrm{sys}(S) = \ell$*, then*

$$|C(S)| \leq 200\,\frac{e^{\ell/2}}{\ell}(2g-2+n).$$

PROOF. If $\ell \leq 2\operatorname{arcsinh} 1$, then all systoles are pairwise disjoint, so

$$|C(S)| \leq \mathrm{Kiss}(S) \leq 3g-3+n.$$

We now suppose that $\ell > 2\operatorname{arcsinh} 1$. Consider $\tilde{S} = S \setminus \bigcup_{c \text{ cusp}} D_{w(\ell)}(c)$, where

$$w(\ell) = \operatorname{arcsinh}\frac{1}{\sinh\frac{\ell}{2}}$$

is the width of a collar around a systole. By the collar lemma, each curve of $C(S)$ is contained in $\tilde{S}$. We cover $\tilde{S}$ with disks of radius $r(\ell)$. Then the cardinality of $C(S)$ is bounded above by

$$\frac{F(S)G(S)}{H(S)},$$

where

$$\begin{aligned} F(S) &= \#\{\text{balls of radius } r(\ell) \text{ needed to cover } \tilde{S}\} \\ G(S) &= \#\{\text{curves in } C(S) \text{ crossing a ball of radius } r(\ell)\} \\ H(S) &= \#\{\text{balls of radius } r(\ell) \text{ a curve in } C(S) \text{ must cross}\} \end{aligned}$$

To bound $|C(S)|$, we need to give upper bounds for $F(S)$ and $G(S)$ and a lower bound for $H(S)$.

Upper bound for $F(S)$

We have

$$F(S) \leq \max \# \left\{\text{embedded balls of radius } \tfrac{r(\ell)}{2} \text{ which are pairwise disjoint}\right\} \leq$$

$$\leq \frac{\mathrm{area}(\tilde{S})}{\mathrm{area}\left(\text{ball of radius } \frac{r(\ell)}{2}\right)} \leq \frac{\mathrm{area}(S)}{2\pi\left(\cosh\frac{r(\ell)}{2} - 1\right)} \leq 8(2g-2+n)e^{\ell/2}.$$

Upper bound for $G(S)$

We proceed as in the proof of Theorem 2.9 in [**Par06**], by reasoning in the universal cover and estimating how many geodesics, pairwise intersecting at an angle of at least θ_ℓ, can intersect a disk of radius $r(\ell)$. We obtain

$$G(S) \leq \frac{\pi}{2}\frac{\sinh(R(\ell) + \operatorname{arcsinh}(1))}{\operatorname{arcsinh}(\sin\theta_\ell)} \leq \frac{5\pi}{2\operatorname{arcsinh}(\sin\theta_\ell)}.$$

Lower bound for $H(S)$

To cover a curve of length ℓ with disks of radius $r(\ell)$ we need at least $\frac{\ell}{2r(\ell)}$. So

$$H(S) \geq \frac{\ell}{2\operatorname{arcsinh}\frac{1}{2\sinh\frac{ell}{4}}} \geq \ell \sinh\frac{\ell}{4}.$$

By putting the three bounds together and considering that $\sinh\frac{\ell}{4}\operatorname{arcsinh}(\sin\theta_\ell)$ is bounded below by $1/3$ for $\ell > 2\operatorname{arcsinh} 1$ we obtain the claimed result. □

2.4. Proof of the new bounds. Using Propositions 6.13, 6.17 and 6.20, we get an upper bound for the kissing number of a surface in terms of its signature and its systole length.

THEOREM 6.21. *If $S \in \mathcal{M}_{g,n}$ ($g \geq 1$, $(g,n) \neq (1,1)$) has systole of length* $\mathrm{sys}(S) = \ell$, *then*

$$\mathrm{Kiss}(S) \leq 20\, n \cosh\frac{\ell}{4} + 200\,\frac{e^{\ell/2}}{\ell}(2g-2+n).$$

As a consequence, we deduce a bound on the kissing number which is independent on the systole length.

THEOREM 6.22. *There exists a universal constant C (which we can take to be 2×10^4) such that for any $S \in \mathcal{M}_{g,n}$, $g \geq 1$, its kissing number satisfies*

$$\mathrm{Kiss}(S) \leq C\,(g+n)\frac{g}{\log(g+1)}.$$

PROOF. It follows from the bounds in Theorem 6.21 and bounds on systole lengths. Precisely we insert the Schmutz Schaller bound (Theorem 5.4) in the term $\cosh\frac{\ell}{4}$ and we use Theorem 5.7 for the $\frac{e^{\ell/2}}{\ell}$ term. For $(g,n) = (1,1)$, we recall the well known fact that $\mathrm{Kiss}(S) \leq 3$ (there can be at most 3 distinct curves that pairwise intersect at most once on a one-holed torus). □

Note that, as in the closed surface case studied by Parlier in [**Par13**], this bound is sub-quadratic in the Euler characteristic.

In the case of punctured spheres, we can get a bound which is linear in the number of cusps.

THEOREM 6.23. *For every $S \in \mathcal{M}_{0,n}$, the number of systoles satisfies*

$$\mathrm{Kiss}(S) \leq \frac{7}{2}n - 5.$$

PROOF. By Proposition 6.13 and Schmutz Schaller's upper bound for the systole, we have

$$|A(S)| \leq \frac{n}{2}\left\lfloor\frac{2(3n-6)}{n}\right\rfloor = \frac{n}{2}\left\lfloor 6 - \frac{12}{n}\right\rfloor \leq \frac{5}{2}n.$$

Moreover, systoles are separating, so can only pairwise intersect an even number of times. This implies that systoles in $\mathfrak{S}(S) \setminus A(S)$ are pairwise disjoint and hence part of a pants decomposition. Note that any pants decomposition of a sphere contains at least two curves bounding two cusps: indeed, the dual graph to the pants decomposition is a tree, so it has at least two leaves, which correspond to curves bounding two cusps. This implies that

$$|\mathfrak{S}(S) \setminus A(S)| \leq \#\text{curves in a pants decomposition} - 2 = n - 5.$$

□

By using short pants decompositions where every curve is of equal length, it is easy to obtain a family of punctured spheres with a number of systoles that grows linearly in the number of cusps. Matching the $\frac{7}{2}n - 5$ upper bound from this theorem seems much more challenging.

Bibliography

[APP11] James W. Anderson, Hugo Parlier, and Alexandra Pettet. Small filling sets of curves on a surface. *Topology Appl.*, 158(1):84–92, 2011.

[Bav96] Christophe Bavard. Disques extrémaux et surfaces modulaires. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):191–202, 1996.

[Bea83] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BM04] Robert Brooks and Eran Makover. Random construction of Riemann surfaces. *J. Differential Geom.*, 68(1):121–157, 2004.

[BS94] Peter Buser and Peter Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, 117(1):27–56, 1994. With an appendix by J. H. Conway and N. J. A. Sloane.

[Bus78] Peter Buser. The collar theorem and examples. *Manuscripta Math.*, 25(4):349–357, 1978.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1992 edition.

[DeB15] Jason DeBlois. The centered dual and the maximal injectivity radius of hyperbolic surfaces. *Geom. Topol.*, 19(2):953–1014, 2015.

[Dia00] Rares Dianu. *Sur le spectre des tores pointés*. PhD thesis. Ecole Polytechnique Fédérale de Lausanne, 2000.

[DP07] Emily B. Dryden and Hugo Parlier. Collars and partitions of hyperbolic cone-surfaces. *Geom. Dedicata*, 127:139–149, 2007.

[Fan15] Federica Fanoni. The maximum injectivity radius of hyperbolic orbifolds. *Geom. Dedicata*, 175:281–307, 2015.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[FN03] Werner Fenchel and Jakob Nielsen. *Discontinuous groups of isometries in the hyperbolic plane*, volume 29 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003. Edited and with a preface by Asmus L. Schmidt, Biography of the authors by Bent Fuglede.

[FP14] Federica Fanoni and Hugo Parlier. Systoles and kissing numbers of finite area hyperbolic surfaces. *To appear in Algebr. Geom. Topol.*, 2014.

[Gen14] M. Gendulphe. Trois applications du lemme de Schwarz aux surfaces hyperboliques. *ArXiv e-prints*, April 2014.

[GS05] Thomas Gauglhofer and Klaus-Dieter Semmler. Trace coordinates of Teichmüller space of Riemann surfaces of signature $(0, 4)$. *Conform. Geom. Dyn.*, 9:46–75, 2005.

[HS85] Joel Hass and Peter Scott. Intersections of curves on surfaces. *Israel J. Math.*, 51(1-2):90–120, 1985.

[HT80] Allen E. Hatcher and William P. Thurston. A presentation for the mapping class group of a closed orientable surface. *Topology*, 19(3):221–237, 1980.

[Kee74] Linda Keen. Collars on Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 263–268. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[KSV07] Mikhail G. Katz, Mary Schaps, and Uzi Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, 76(3):399–422, 2007.

[MP08] Greg McShane and Hugo Parlier. Multiplicities of simple closed geodesics and hypersurfaces in Teichmüller space. *Geom. Topol.*, 12(4):1883–1919, 2008.

[MRT14] Justin Malestein, Igor Rivin, and Louis Theran. Topological designs. *Geom. Dedicata*, 168:221–233, 2014.

[Par06] Hugo Parlier. Hyperbolic polygons and simple closed geodesics. *Enseign. Math. (2)*, 52(3-4):295–317, 2006.

[Par13] Hugo Parlier. Kissing numbers for surfaces. *J. Topol.*, 6(3):777–791, 2013.

[Par14] Hugo Parlier. Simple closed geodesics and the study of Teichmüller spaces. In *Handbook of Teichmüller theory. Volume IV*, volume 19 of *IRMA Lect. Math. Theor. Phys.*, pages 113–134. Eur. Math. Soc., Zürich, 2014.

[Pet13] Bram Petri. Random regular graphs and the systole of a random surface. *ArXiv e-prints*, November 2013.

[Prz15] Piotr Przytycki. Arcs intersecting at most once. *Geom. Funct. Anal.*, 25(2):658–670, 2015.

[Ran79] Burton Randol. Cylinders in Riemann surfaces. *Comment. Math. Helv.*, 54(1):1–5, 1979.

[Sch94] Paul Schmutz. Congruence subgroups and maximal Riemann surfaces. *J. Geom. Anal.*, 4(2):207–218, 1994.

[Sch96] Paul Schmutz. Arithmetic Fuchsian groups and the number of systoles. *Math. Z.*, 223(1):13–25, 1996.

[SS97] Paul Schmutz Schaller. Extremal Riemann surfaces with a large number of systoles. In *Extremal Riemann surfaces (San Francisco, CA, 1995)*, volume 201 of *Contemp. Math.*, pages 9–19. Amer. Math. Soc., Providence, RI, 1997.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[Thu98] William P. Thurston. Minimal stretch maps between hyperbolic surfaces. *ArXiv Mathematics e-prints*, January 1998.

[TWZ06] Ser Peow Tan, Yan Loi Wong, and Ying Zhang. Generalizations of McShane's identity to hyperbolic cone-surfaces. *J. Differential Geom.*, 72(1):73–112, 2006.

[Yam82] Akira Yamada. On Marden's universal constant of Fuchsian groups. II. *J. Analyse Math.*, 41:234–248, 1982.

Curriculum Vitae

Personal information

Place and date of birth: Sondrio, September 12th, 1986

Nationality: Italian

Languages: Italian (mother tongue), English and French (fluent), Dutch and German (basic)

Education

Bachelor in Mathematics, University of Milan, Italy (September 2005–July 2008)

Master in Mathematics, University of Milan, Italy (September 2008–September 2010)

Doctoral student in Mathematics, University of Fribourg, Switzerland (January 2011–June 2015)

Research interests

Hyperbolic surfaces, two-dimensional hyperbolic orbifolds, Teichmüller and moduli spaces

Research stays

Erwin Schrödinger International Institute for Mathematical Physics, Austria (March 2013)

CUNY Graduate Center, United States of America (January–June 2015)