Solutions to some exercises

Federica Fanoni
federica.fanoni@u-pec.fr

September 30, 2021

Exercise 1.8
Prove that the subspace topology is a topology.

Solution: let (X,τ) be a topological space, $Y \subset X$ and τ_Y the subspace topology. Let us verify the three conditions for a collection of subsets to be a topology:

1. $\emptyset = \emptyset \cap Y \in \tau_Y$, since $\emptyset \in \tau$. Similarly, $Y = X \cap Y \in \tau_Y$, since $X \in \tau$;

2. let $U_i, i \in I$ be a collection of open sets of Y. Then, by definition of the subspace topology, for every $i \in I$ there is $V_i \in \tau$ such that $U_i = V_i \cap Y$. So
$$\bigcup_{i\in I} U_i = \bigcup_{i\in I}(V_i \cap Y) = \left(\bigcup_{i\in I} V_i\right) \cap Y \in \tau_Y,$$
since $\bigcup_{i\in I} V_i \in \tau$ (τ is a topology);

3. let $U_1, \ldots, U_n$ be a finite collection of open sets of Y. As before, for every $i = 1, \ldots, n$ there is $V_i \in \tau$ such that $U_i = V_i \cap Y$. Then
$$\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n}(V_i \cap Y) = \left(\bigcap_{i=1}^{n} V_i\right) \cap Y$$
and since τ is a topology, $\bigcap_{i=1}^{n} V_i \in \tau$ and thus $\bigcap_{i=1}^{n} U_i \in \tau_Y$, by definition of the subspace topology.

Exercise 1.21
Show that $S = [-1,1] \times [-1,1]$ and $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ are homeomorphic.

Solution: note that
$$D = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\}$$
and
$$S = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\|_\infty \leq 1\},$$

where $\|(x,y)\|_\infty = \max\{|x|,|y|\}$. So we define the function

$$\begin{aligned} f : S &\to D \\ (x,y) &\mapsto \frac{\|(x,y)\|_\infty}{\|(x,y)\|}(x,y) \qquad \text{if } (x,y) \neq (0,0) \\ (0,0) &\mapsto (0,0) \end{aligned}$$

which sends $v = (x,y)$ to the unique point (x',y') on the oriented ray from $(0,0)$ and through v satisfying $\|(x',y')\|_\infty = \|(x,y)\|$.

Continuity is shown as in any analysis course. Moreover, there is an obvious inverse function, which can also be shown to be continuous in the same way:

$$\begin{aligned} g : D &\to S \\ (x,y) &\mapsto \frac{\|(x,y)\|}{\|(x,y)\|_\infty}(x,y) \qquad \text{if } (x,y) \neq (0,0) \\ (0,0) &\mapsto (0,0) \end{aligned}$$

so f is a homeomorphism.

Lemma
Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$ a continuous function. If $Z \subset X$ is endowed with the subspace topology, then

$$f|_Z : Z \to Y$$

is a continuous function.

Proof. Let $U \in \sigma$. Then $(f|_Z)^{-1}(U) = f^{-1}(U) \cap Z$; since f is continuous, $f^{-1}(U) \in \tau$, so $(f|_Z)^{-1}(U)$ is open in Z (by definition of the subspace topology). □

Example 2.14
Let $X = [0,1] \times [0,1]$ and $\sim$ the equivalence relation defined in example 2.14. Then $T = X/_\sim$ is a surface.

Partial solution: we want to find, for any $P = [(x_0,y_0)] \in T$, an open neighborhood of P homeomorphic to an open set in $\mathbb{R}^2$. There are three possibilities for P: either (x_0,y_0) does not belong to a side of X, or it belongs to a side, but not to a corner, or it is a corner. We will give the details of the construction of an open neighborhood and of a homeomorphism from the neighborhood to an open set of $\mathbb{R}^2$ only in the first two cases. The idea for the third case is similar to the one for the second case, but it becomes even more annoying to write down.
Notation: $\mathring{X} =]0,1[\times]0,1[$.

An observation we will use throughout is the following: if $(x,y) \in \mathring{X}$, $\pi((x,y)) = [(x,y)] = \{(x,y)$. In particular, π is injective on $\mathring{X}$.
Case 1: $(x_0,y_0) \in \mathring{X}$. Then there is $\varepsilon > 0$ such that $B := B_\varepsilon((x_0,y_0)) \subset \mathring{X}$. Define then

$$U_P := \pi(B).$$

Since π is injective on $\mathring{X}$,

$$\pi^{-1}(U_P) = B,$$

so by definition of the quotient topology U_P is open in T (as B is open in X). Moreover, $P \in U_P$, so U_P is an open neighborhood of P.

Define $f := \pi|_B : B \to U_P$. We now show that f is a homeomorphism, so that $f^{-1} : U_P \to B$ is the required homeomorphism between an open neighborhood of P and an open set of $\mathbb{R}^2$.

1. f is bijective, because:
   - π is injective on $\mathring{X}$, so f is injective;
   - U_P is defined to be the image of B via f, so f is surjective.
2. π is a continuous function and f is the restriction of a continuous function, so by the lemma above f is continuous.
3. f is open: let $U \subset B$ be an open set; we need to show that $f(U$ is open, i.e. (by the definition of the quotient topology) that $\pi^{-1}(f(U))$ is open in X. If we show that $\pi^{-1}(f(U)) = U$, we are done, since U is open in X. Let us show that, by proving the double inclusion:
   - if $(x, y) \in U$, by the definition of f we have that $\pi(x, y) = f(x, y) \in f(U)$, i.e. $(x, y) \in \pi^{-1}(f(U))$, thus $U \subset \pi^{-1}(f(U))$;
   - if $(x, y) \in \pi^{-1}(f(U))$, then $\pi(x, y) \in f(U)$, i.e. there is $(\bar{x}, \bar{y}) \in U$ such that $\pi(x, y) = f(\bar{x}, \bar{y}) = \pi(\bar{x}, \bar{y})$. By the observation above, this means that $(x, y) = ((\bar{x}, \bar{y}) \in U$. So $\pi^{-1}(f(U)) \subset U$.

So f is an open continuous bijection and thus it is a homeomorphism.
<u>Case 2:</u> (x_0, y_0) belongs to a side of X, but it is not a corner. For simplicity, let us assume then $x_0 = 0$ and $y_0 \neq 0$. The other cases are analogous.

In this case

$$[(x_0, y_0)] = \{(0, y_0), (1, y_0)\}.$$

Let $\varepsilon > 0$ be small enough so that $B_\varepsilon((0, y_0))$ does not contain any corner of X. Note that in particular $\varepsilon \leq \frac{1}{2}$. Define

$$V_P = \pi(B_\varepsilon((0, y_0)) \cap X) \cup \pi(B_\varepsilon(1, y_0) \cap X) = \pi(B_\varepsilon((0, y_0)) \cap X) \cup \pi(B_\varepsilon(1, y_0) \cap \mathring{X})$$

and let $g : B_\varepsilon((0, y_0)) \to V_P$ be the map

$$g(x, y) = \begin{cases} \pi(x, y) & \text{if } (x, y) \in B_\varepsilon((0, y_0)) \cap X \\ \pi(x + 1, y) & \text{if } (x, y) \in B_\varepsilon((0, y_0)) \setminus X \end{cases}$$

V_P is open because

$$\pi^{-1}(V_P) = (B_\varepsilon((0, y_0)) \cap X) \cup (B_\varepsilon((1, y_0)) \cap X) = (B_\varepsilon((0, y_0)) \cup B_\varepsilon((1, y_0))) \cap X$$

which is open in X, since it's the intersection of an open set of $\mathbb{R}^2$ with X. So by definition of the quotient topology V_P is open.

Let us show that g is a homeomorphism.

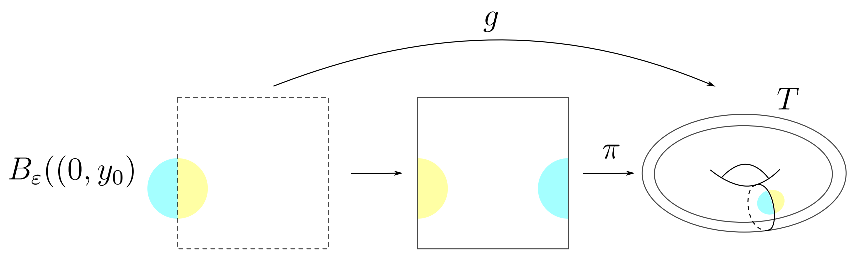

Figure 1: A depiction of the map g

1. g is a bijection because:

   - suppose $g(x_1, y_1) = g(x_2, y_2)$. If $(x_1, y_1), (x_2, y_2) \in B_\varepsilon((0, y_0)) \cap X$, then $g(x_i, y_i) = \pi(x_i, y_i)$, so $g(x_1, y_1) = g(x_2, y_2)$ means $\pi(x_1, y_1) = \pi(x_2, y_2)$. But no two distinct points in $B_\varepsilon((0, y_0)) \cap X$ are equivalent under $\sim$, so $(x_1, y_1) = (x_2, y_2)$. If $(x_1, y_1) \in B_\varepsilon((0, y_0)) \cap X$ and $(x_2, y_2) \in B_\varepsilon((0, y_0)) \setminus X$, then $\pi(x_1, y_1) = g(x_1, y_1) = g(x_2, y_2) = \pi(x_2 + 1, y_2)$. By the conditions on ε and (x_2, y_2), $(x_2 + 1, y_2) \in \mathring{X}$, so $\pi(x_2 + 1, y_2) = \{(x_2 + 1, y_2)\}$ and thus $(x_1, y_1) = (x_2 + 1, y_2)$. But by the conditions on the x_i and ε, $x_1 \neq x_2 + 1$, so we get a contradiction. We proceed similarly for the cases $(x_2, y_2) \in B_\varepsilon((0, y_0)) \cap X$, $(x_1, y_1) \in B_\varepsilon((0, y_0)) \setminus X$ and $(x_1, y_1), (x_2, y_2) \in B_\varepsilon((0, y_0)) \setminus X$. So g is injective.
   - let $Q =\in V_P$. If $Q \in \pi(B_\varepsilon(0, y0) \cap X$, there is $(x, y) \in B_\varepsilon(0, y_0) \cap X)$ such that
     $$Q = \pi(x, y) = g(x, y).$$
     If instead $Q \in \pi(B_\varepsilon(1, y_0) \cap \mathring{X})$, there is $(x, y) \in B_\varepsilon(1, y_0) \cap \mathring{X}$ such that $Q = \pi(x, y)$. Then $(x - 1, y) \in B_\varepsilon(0, y_0) \setminus X$ and
     $$g(x - 1, y) = \pi((x - 1) + 1, y) = \pi(x, y) = Q.$$
     In both cases, Q is the image of an element in $B_\varepsilon(0, y_0)$, i.e. g is surjective.

2. g is continuous: let $U \subset V_P$ be an open set, i.e. $\pi^{-1}(U) \subset X$ is open. To show that $g^{-1}(U)$ is open, we prove that given any point $(x, y) \in g^{-1}(U)$ there is $\delta > 0$ such that $B_\delta(x, y) \subset g^{-1}(U)$. We have three cases: $x > 0$, $x = 0$ and $x < 0$.

   - if $x > 0$, then $g(x, y) = \pi(x, y)$ and
     $$\pi^{-1}(\pi(x, y)) = \{(x, y)\} \subset \pi^{-1}(U) \cap B_\varepsilon(0, y_0).$$
     Since $\pi^{-1}(U) \cap B_\varepsilon(0, y_0)$ is open in R^2 (intersection of two open sets), there is $\delta > 0$ such that
     $$B_\delta(x, y) \subset \pi^{-1}(U) \cap B_\varepsilon(0, y_0),$$
     so
     $$B_\delta(x, y) = g^{-1}(\pi(B_\delta(x, y))) \subset g^{-1}(U).$$

- if $x < 0$, we argue similarly to the case $x > 0$.
- if $x = 0$, then $g(0, y) = [(0, y)]$ and

  $$\pi^{-1}([0, y]) = \{(0, y), (1, y)\}.$$

  We have:

  $$(0, y) \in \underbrace{\pi^{-1}(U)}_{\text{open}} \Rightarrow \exists \delta_1 > 0 : B_{\delta_1}(0, y) \subset \pi^{-1}(U)$$
  $$\Rightarrow B_{\delta_1}(0, y) \cap X \subset g^{-1}(U)$$

  and

  $$(1, y) \in \underbrace{\pi^{-1}(U)}_{\text{open}} \Rightarrow \exists \delta_2 > 0 : B_{\delta_2}(0, y) \subset \pi^{-1}(U)$$
  $$\Rightarrow B_{\delta_2}(0, y) \setminus X \subset g^{-1}(U).$$

  Thus

  $$B_\delta(0, y) \subset g^{-1}(U)$$

  for $\delta = \min\{\delta_1, \delta_2\}$.

3. g is open: suppose $U \subset B_\varepsilon(1, y_0)$ is open. Note that U is open in $\mathbb{R}^2$ too and

   $$U = U^+ \cup U^-$$

   where $U^+ = \{(x, y) \in U \mid x \geq 0\}$ and $U^- = \{(x, y) \in U \mid x \leq 0\}$. Note that

   $$U^+ \cap U^- = U^0,$$

   where $U^0 = \{(x, y) \in U \mid x = 0\}$. Now

   $$g(U) = g(U^+) \cup g(U^-) = \pi(U^+) \cup \pi(U^- + (1, 0)),$$

   where given a set $A \subset \mathbb{R}^2$, we define

   $$A + (1, 0) = \{(x + 1, y) \mid (x, y) \in A\}.$$

   Moreover $\pi^{-1}(g(U)) = \pi^{-1}(g(U^+)) \cup \pi^{-1}(g(U^-))$. We have:

   $$\pi^{-1}(g(U^+)) = U^+ \cup (U^0 + (1, 0)) \text{ and } \pi^{-1}(g(U^-)) = (U^- + (1, 0)) \cup U^0$$

   so

   $$\pi^{-1}(g(U)) = U^+ \cup (U^0 + (1, 0)) \cup (U^- + (1, 0)) \cup U^0 = U^+ \cup (U^- + (1, 0)).$$

   Since $U^+ = U \cap X$, U^+ is open in X, and so is $U^- + (1, 0) = (U + (1, 0) \cap X)$ (since $U + (1, 0)$ is open in R^2). Thus $\pi^{-1}(g(U))$ is a union of two open set, hence open. So $g(U)$ is open in T.

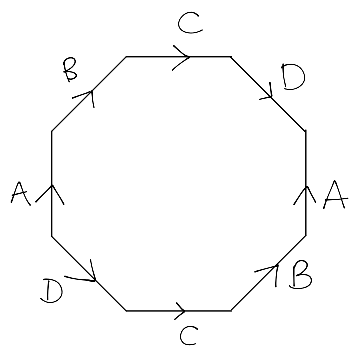

Figure 2: An octagon with opposite sides identified

As g is a continuous open bijection, it is a homeomorphism and so

$$g^{-1} : V_P \to B_\varepsilon(0, y_0)$$

is also a homeomorphism.

Exercise 3.17
Verify that the surface in Figure 2 is homeomorphic to S_2.

Solution: since S is the quotient of a single polygon, it is path-connected and compact. If we look at Figure 3, the green arrows indicate the orientation induced on the sides by the counterclockwise orientation of the boundary of the octagon. We can then check that for every pair of sides with the same labels, exactly one of the two sides has an arrow giving the same orientation as the green arrow. So S is orientable. So by the classification theorem of surfaces we know that S is

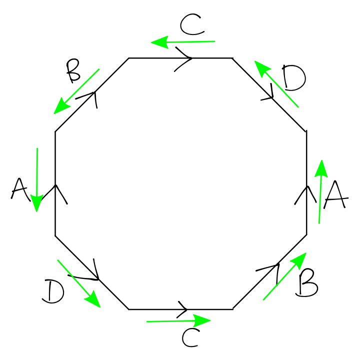

Figure 3: Checking orientability

homeomorphic to S_g, for some $g \geq 0$, and thus $\chi(S) = \chi(S_g) = 2 - 2g$. Let us compute $\chi(S)$, to be able to compute g. We have (see Figure 4:

- $f = 1$ because we have only one polygon;
- $e = \frac{8}{2} = 4$;

- $v = 1$ because: $a \sim b$, since they are both the starting point of a side labelled A, $b \sim c$ because they are both the endpoint of a side labelled B, $c \sim d$ (starting point of C), $d \sim e$ (endpoints of D), $e \sim f$ (endpoints of A), $f \sim g$ (starting points of B) and $g \sim h$ (endpoints of C). So all vertices are equivalent, i.e. there is a single equivalence class of vertices.

Thus

$$\chi(S) = 1 - 4 + 1 = -2$$

and hence $g = 2$: S is homeomorphic to S_2.

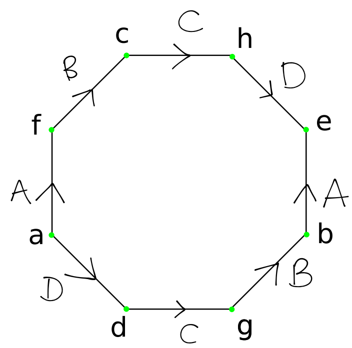

Figure 4: Computing the Euler characteristic