Exam

Basics in Mathematics – Geometry

Duration: 30 minutes

Exercise 1. Let $X = [-2,-1] \cup [1,2]$ and $\sim$ the equivalence relation given by $x \sim y$ if $xy > 0$. Describe the equivalence classes of $\sim$.

Solution. Note first that $x \sim y$ if and only if either

$$\begin{cases} x > 0 \\ y > 0 \end{cases}$$

or

$$\begin{cases} x < 0 \\ y < 0 \end{cases}$$

So if $x \in X$ and $x > 0$, its equivalence class is

$$[x] = \{y \in X \mid y > 0\} = [1,2]$$

and if if $x \in X$ and $x < 0$, its equivalence class is

$$[x] = \{y \in X \mid y < 0\} = [-2,-1],$$

i.e. there are two equivalence classes, $[1,2]$ and $[-2,-1]$.

Exercise 2. Let $(X,\tau),(Y,\sigma),(Z,\eta)$ be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Prove that $g \circ f : X \to Z$ is continuous.

Solution. We need to show that for every $U \in \eta$, $(g \circ f)^{-1}(U) \in \tau$. We have

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

and since g is continuous, $g^{-1}(U) \in \sigma$. Moreover f is continuous, so $f^{-1}(V) \in \tau$ for every $V \in \sigma$. In particular $f^{-1}(g^{-1}(U)) \in \tau$.

Exercise 3. Consider the surface S in Figure 1.

1. Is S orientable? Why?
2. Compute the Euler characteristic $\chi(S)$ of S.

Solution.

1. S is not orientable: in Figure 2, we indicate with the green arrows the orientation induced on the edges by the counterclockwise orientation of the boundary of the polygon. Then they disagree with the orientation given by the side identifications for *both* sides labelled C. By the proposition seen in class, this implies that S is not orientable.
   Alternatively, in Figure 2 we have highlighted an embedded Möbius band.

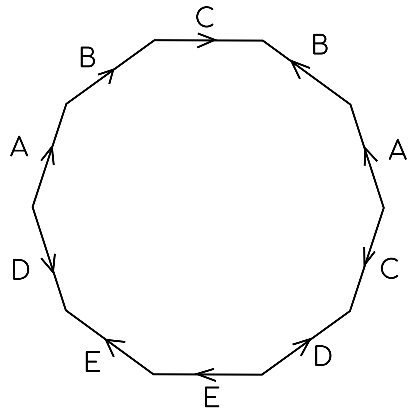

Figure 1: The surface S

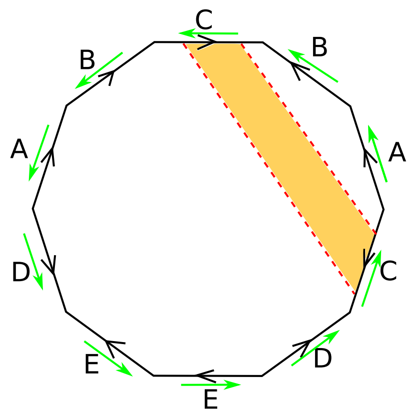

Figure 2: The surface S is not orientable

2. The Euler characteristic of S is given by $\chi(S) = v - e + f$, where

   - v is the number of equivalence classes of vertices of the polygon on the surface;
   - $e = \frac{10}{2} = 5$ is half the total number of edges of the polygon;
   - $f = 1$ is the number of polygons.

   To compute v, let us look at which vertices are equivalent to each other (see Figure 3):

   - $p_1 \sim p_2$ (both the start of a side labeled A), $p_2 \sim p_3$ (both the start of a side labeled C), $p_3 \sim p_4$ (both the end of a side labeled B), $p_4 \sim p_5$ (both the end of a side labeled C), $p_5 \sim p_6$ (both the end of a side labeled D), $p_6 \sim p_7$ (both the end of a side labeled E) and $p_8 \sim p_8$ (both the start of a side labeled E);
   - $p_9 \sim p_{10}$ (both the end of a side labeled B).

   So $v = 2$ and
$$\chi(S) = 2 - 5 + 1 = -2.$$

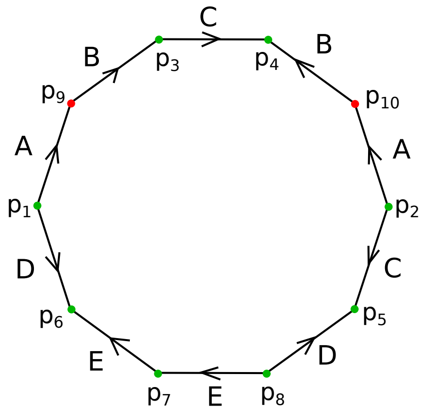

Figure 3: Computing v

Exercise 4. Consider the surfaces X and Y in Figure 4. Are they path-connected? (No justification needed.)

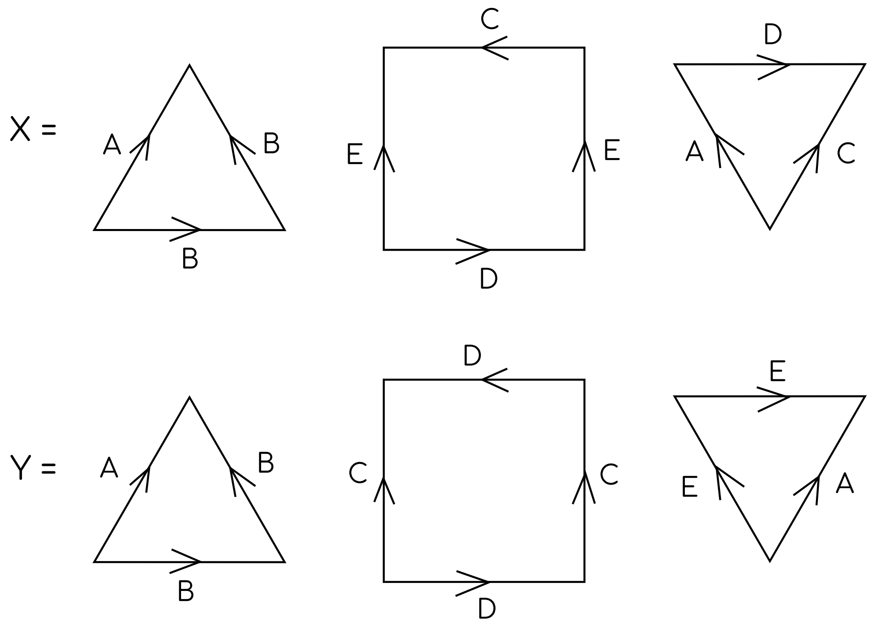

Figure 4: The surfaces X and Y

Solution. The surface X is path-connected, while the surface Y isn't.