Basics in Mathematics – Geometry

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1 Lecture 1

1.1 Surfaces: an informal introduction

Our first idea of a surface is probably *something two-dimensional*. The obvious example of a two-dimensional mathematical object is the plane $\mathbb{R}^2$. We want to say that a surface is something that, sufficiently close to any point, looks like the plane. Let's imagine being in the middle of the ocean: since we are so small compared to the size of the ocean, we see such a small portion that it looks like a plane to us. It's the same that (probably) happens to an ant on a balloon: it does not notice that it curves, nor that globally it's not a plane at all.

Our first objective is to formalize the idea of a surface as something that, sufficiently close to any point, looks like the plane. We need to formalize the concepts of *sufficiently close* and *looking like*. To speak about *sufficiently close* it would be enough to speak about a way to measure distances and say that sufficiently close to a point means at distance bounded by some very small number. But for the *looking like* part, if we think of a piece of sphere and a piece of plane, since one is curved and the other is flat, we need to *deform* the piece of the sphere to make it coincide to a piece of the plane. The key concept here will be that of *continuous deformations* – intuitively, deformations which not involve cutting or gluing. The correct setup to describe such deformations is that of a branch of geometry called *topology*. The first part of this course will be a quick introduction to this area of mathematics.

1.2 Point-set topology

Definition 1.1. Let X be a set. A *topology* on X is a collection τ of subsets of X ($\tau \subset \mathcal{P}(X)$) such that:

1. $\emptyset, X \in \tau$;
2. the union of any collection of sets in τ is in τ (i.e. for every set of indices I and for every $U_i \in \tau, i \in I$, $\bigcup_{i\in I} U_i \in \tau$);

3. the intersection of any *finite* collection of sets in τ is in τ (i.e. for every integer $n \geq 1$, for every $U_1, \ldots, U_n \in \tau$, $\bigcap_{i=1}^{n} U_i \in \tau$).

Elements of τ are called *open* sets. The pair (X, τ) is called *topological space.*

Definition 1.2. Let (X, τ) be a topological space. A set C is *closed* if its complement $X \setminus C$ is open. For $x \in X$, an *open neighborhood* of x is an open set containing x.

Remark 1.3. It follows from the definition of topology and De Morgan's laws (the way union, intersection and taking the complement interact) that

1. $\emptyset$ and X are closed sets;
2. the intersection of an arbitrary collection of closed sets is closed;
3. a finite union of closed sets is closed.

Note in particular that being open and being closed are not mutually exclusive: for instance, $\emptyset$ and X are open and closed.

Let us look at some examples.

Example 1.4. You have probably already seen the definition of open set in $\mathbb{R}^n$: a subset $U \subset \mathbb{R}^n$ is open if for every $x \in U$ there is $r > 0$ (depending on x) such that $B_r(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\} \subset U$. Here $\| \cdot \|$ denotes the standard norm on $\mathbb{R}^n$ ($\|y\| = \sqrt{\sum_{i=1}^{n} y_i^2}$), so that $B_r(x)$ is the open ball centered in x and of radius r. Let us show that the collection of open sets is a topology, by proving it satisfies the three properties of Definition 1.1:

1. $\emptyset$ is open, because the condition is trivially satisfied, and $\mathbb{R}^n$ is open because for every $x \in \mathbb{R}^n$ and for every $r > 0$, $B_r(x) \subset \mathbb{R}^n$;
2. if $U_i, i \in I$, is a collection of open sets and $x \in \bigcup_{i \in I} U_i$, then there is $i \in I$ such that $x \in U_i$. As U_i is open, there is $r > 0$ such that $B_r(x) \subset U_i$ and thus $B_r(x) \subset \bigcup_{i \in I} U_i$;
3. let $U_1, \ldots, U_k$ be a finite collection of open set and let $x \in \bigcap_{i=1}^{k} U_i$. For every i there is $r_i > 0$ such that $B_{r_i}(x) \subset U_i$. Then $r := \min\{r_i \mid i = 1, \ldots, k\} > 0$ and $B_r(x) \subset U_i$ for every i, that is, $B_r(x) \subset \bigcap_{i=1}^{k} U_i$.

Note that the intersection of an arbitrary family of open sets is not necessarily open: for instance, for every $n \in \mathbb{N}$, $B_{1/n}(x)$ is an open set, but

$$\bigcap_{n \in \mathbb{N}} B_{1/n}(x) = \{x\}$$

and $\{x\}$ is not open.

Recall that points are closed subsets of $\mathbb{R}^n$.

Example 1.5. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then τ is a topology on X. Indeed:

1. $\emptyset, X \in \tau$;

2. if $U_i \in \tau$, $i \in I$, for some set of indices I, then $\bigcup_{i\in I} U_i$ is either X, if one of the U_i is X, or $\{a\}$, if no U_i is X and some U_i is $\{a\}$, or $\emptyset$ otherwise. In any case, the union is in τ;

3. similarly, we can check that any finite intersection is in τ.

Example 1.6. If X is a singleton, $X = \{p\}$, there is only one possible topology on it: $\tau = \{\emptyset, X\}$, because these two sets need to be part of any topology and there are no other subsets.

Usually, we will consider $\mathbb{R}^n$ with the topology described in Example 1.4, in which case we will usually not specify the topology. But it is important to notice that a topological space is really a pair, given by a set and a topology on it. For instance, we could give a completely different topology to $\mathbb{R}^n$: we could consider $\sigma = \{\emptyset, \mathbb{R}^n\}$.

Often we are given a topological space and we want to consider one of its subset as a topological space in itself ("forgetting" that it was a subset). Here is how we do it:

Definition 1.7. Let (X, τ) be a topological space and Y a subset of X. The *subspace topology* on Y is the collection $\tau_Y := \{U \cap Y \mid U \in \tau\}$.

Exercise 1.8. Prove that the subspace topology is a topology.

Example 1.9. Let $Y = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. Both

$$U = \left\{ x \in Y \mid \|x\| < \frac{1}{2} \right\}$$

and

$$V = [-1, 0[\times[-1, 1]$$

are elements of τ_Y (we say that they are *open as subsets of* Y), because

$$U = Y \cap B_{1/2}(0)$$

and

$$V = Y \cap (]-\infty, 0[\times\mathbb{R})$$

and $B_{1/2}(0),]-\infty, 0[\times\mathbb{R}$ are open sets of $\mathbb{R}^2$. Note though that while U is an open set of $\mathbb{R}^2$ as well, V is not!

Exercise 1.10. Is there a necessary and sufficient condition on $Y \subset X$ so that $\tau_Y \subset \tau$ (i.e. every set which is open in Y is open in X as well?

Another simple operation to produce new topological spaces from old ones is to take disjoint unions:

Definition 1.11. Let (X_1, τ_1) and (X_2, τ_2) be two topological space. Their *disjoint union* is the topological space (Y, σ) given by

$$Y = X_1 \sqcup X_2$$

and

$$\sigma = \{U \subset X_1 \sqcup X_2 \mid U \cap X_1 \in \tau_1, U \cap X_2 \in \tau_2\}.$$

Example 1.12. $\mathbb{R} \sqcup \mathbb{R}$ is the topological space

$$X = \mathbb{R} \sqcup \mathbb{R} = \{(p, i) \mid p \in \mathbb{R}, i = 1, 2\}$$

with open sets $U \in X$ such that $\{p \mid (p, 1) \in U\}$ and $\{p \mid (p, 2) \in U\}$ are open in $\mathbb{R}$. Graphically, we can think of it[1] as two disjoint lines in the plane.

Similarly, we can consider $\mathbb{R} \sqcup \{p\}$; here open sets are either of the form U, where $U \subset \mathbb{R}$ is open, or $U \cup \{p\}$, where again U is an open subset of $\mathbb{R}$. As before, we can think of this space as a line and a point disjoint from it in the plane.

With the notion of topology, we are now able to formalize the concept of *sufficiently close*, by saying *in a (small enough) open neighborhood* of a point. The next concept we need to formalize is that of *looking like*. Essentially, we will want a notion that identifies two topological spaces if the sets are in bijection and this bijection respects the topologies (gives a bijection between open sets). This will be given by the existence of a continuous function with continuous inverse. So we first need to define what a continuous function is. We already have the concept of continuity for functions $f : \mathbb{R}^n \to \mathbb{R}^m$:

Definition 1.13. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if for every $x_0 \in \mathbb{R}^n$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathbb{R}^n$ with $\|x - x_0\| < \delta$, we have $\|f(x) - f(x_0)\| < \varepsilon$.

But topological spaces don't have a norm (or more generally a notion of distance). So we want to try to find an equivalent definition of continuity which can be expressed in terms of open sets.

So note that if $f(x_0) = y_0$, we can rephrase the notion of continuity (at x_0) by saying that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f(B_\delta(x_0)) \subset B_\varepsilon(y_0).$$

Now, if U is an open set of $\mathbb{R}^m$, for every $y \in U$ there is $\varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \subset U$. If $x \in \mathbb{R}^n$ is such that $f(x) = y$, then we have seen that there is $\delta_x > 0$ such that

$$f(B_{\delta_x}(x)) \subset B_{\varepsilon_y}(y) \subset U,$$

or

$$B_{\delta_x}(x) \subset f^{-1}(U).$$

In particular this shows that if U is open, every point in $f^{-1}(U)$ has a small open ball around it which is contained in $f^{-1}(U)$, i.e. $f^{-1}(U)$ is open.

In fact one can show that this condition characterizes continuity:

[1] I.e. it looks like – we will formalize this concept soon.

Proposition 1.14. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous (in the sense of Definition 1.13) if and only if for every $U \subset \mathbb{R}^m$ open, $f^{-1}(U)$ is open.*

So, for general topological spaces, we can define continuity as follows, as a generalization of what happens in $\mathbb{R}^n$:

Definition 1.15. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : X \to Y$ is *continuous* if for every $V \in \sigma$, $f^{-1}(V) \in \tau$ (in words: if the preimage of any open set is an open set).

To say that two topologically spaces are "topologically" the same, we want to say that one can be continuously deformed into the other and vice versa. We could formalize the concept of continuously deforming one object into another by saying that there is a continuous map from one to the other, but we also want a way to go back, to reverse the deformation. This leads us to the concept of *homeomorphism*:

Definition 1.16. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : X \to Y$ is a *homeomorphism* if it is continuous, bijective and its inverse is continuous. Two topological spaces are *homeomorphic* if there is a homeomorphism between them.

Note that if $f : X \to Y$ is a bijection, f^{-1} is continuous if (by definition) for every open $U \subset X$, $(f^{-1})^{-1}(U) = f(U)$ is open.

Definition 1.17. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : X \to Y$ is *open* if for every $U \in \tau$, $f(U) \in \sigma$.

Using this definition and the observation before, we can also say that a homeomorphism is a continuous and open bijective map.

Let us look at some other topological spaces, the spheres:

Definition 1.18. The *n-sphere* is the topological space

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

(with the subspace topology).

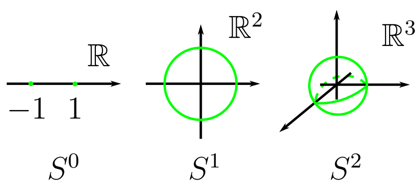

Figure 1: Spheres of dimension $0, 1$ and 2

We can show what we have informally discussed before: the disjoint union of two topological spaces looks like the two topological spaces disjointly embedded in a bigger space (when this is possible). For instance, we can show:

Example 1.19. S^0 is homeomorphic to the disjoint union $X \sqcup Y$, where $X = \{a\}$ and $Y = \{b\}$ are both singletons. There are two obvious bijections; we choose one (but either would do):

$$\begin{aligned} f : S^0 &\to X \sqcup Y \\ -1 &\mapsto a \\ 1 &\mapsto b \end{aligned}$$

We want to show that f is a homeomorphism. Let's first show that it's continuous: open sets in $X \sqcup Y$ are $\emptyset$, X, Y and $X \sqcup Y$ (since, as mentioned before, there are only two open sets each in X and Y). Then

- $f^{-1}(\emptyset) = \emptyset$, which is open in S^0;
- $f^{-1}(X \sqcup Y) = S^0$, which is open in S^0;
- $f^{-1}(X) = \{-1\} = S^0 \cap]0, 2[$, so it is open in S^0, by definition of the subspace topology, and similarly $f^{-1}(Y) = \{1\}$ is open.

Similarly we can check that f is open: by what we have seen, the open sets in S^0 are $\emptyset, S^0, \{-1\}$ and $\{1\}$ and we can explicitly compute the images of all of them.

Note that S^0 gives us an example of a space in which proper (i.e. different from $\emptyset$ and from the full set) subsets are open and closed at the same time: both its points are open and closed.

Exercise 1.20. Formalize the fact that $\mathbb{R} \sqcup \{p\}$ is a line and a point in the plane: show that it is homeomorphic to

$$X = \{(x, 0) \mid x \in \mathbb{R}\} \cup \{(0, 1)\} \subset \mathbb{R}^2$$

with the subspace topology.

Exercise 1.21. Show that $S = [-1, 1] \times [-1, 1]$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ are homeorphic.

Here's an informal interpretation of this last example, which might give you an intuition of what a homeomorphism here: if we think of the square S and the disk D as made of rubber, we can stretch the disk onto the square and shrink the square onto the disk. This is a bit the idea of what a homeomorphism does: it stretches, but it does not tear (we cannot use scissors or poke a hole into our piece of rubber).

Exercise 1.22. Let (X, τ) be a topological space. Let $Y \subset X$. Prove that the inclusion map $Y \hookrightarrow X$ is continuous, if Y has the subspace topology.

The concept of homeomorphism formalizes then the idea of *looking like*. So we have all the tools necessary for giving a mathematical definition of a surface, which we will do in the next lecture.

2 Lecture 2

2.1 Surfaces: a formal introduction

Definition 2.1. Let (X, τ) be a topological space. X is a *surface* if it is locally homeomorphic to $\mathbb{R}^2$, i.e. for every $x \in X$ there is an open neighborhood U of x and a homeomorphism between U and an open set of $\mathbb{R}^2$.

Actually, there are two technical conditions which need to be added to the definition of surface: we require a surface to also be *second countable* and *Hausdorff*. In practice, most "reasonable" topological spaces will satisfy these conditions, and in particular all the ones we will consider, so we will forget about these technical hypotheses.

We can replace, in the definition of surface, $\mathbb{R}^2$ by $\mathbb{R}^n$, and describe spaces which are n-dimensional:

Definition 2.2. A *manifold of dimension n* (or *n-manifold*) is a (second countable, Hausdorff) topological space which is locally homeomorphic to $\mathbb{R}$.

The most basic example of an n-manifold is $\mathbb{R}^n$. More interesting examples are the spheres, in all dimensions.

Example 2.3. The 2-sphere is the subspace of $\mathbb{R}^3$ given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The North pole is the point $N = (0, 0, 1)$; the *stereographic projection from the North pole* is the map

$$\begin{aligned} \varphi : S^2 \smallsetminus \{N\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \end{aligned}$$

Given a point $P = (x, y, z) \neq N$ on the sphere, there is a unique line through N and P, which intersects the plane $z = 0$ in $\mathbb{R}^3$ in a unique point $(a, b, 0)$. One can check that $\varphi(P) = (a, b)$.

It is possible to show that φ is actually a homeomorphism from $S^2 \smallsetminus \{N\}$ to $\mathbb{R}^2$, the inverse map being given by

$$\varphi^{-1}(X, Y) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}.\right)$$

Also, $S^2 \smallsetminus \{N\}$ is an open set of the sphere ($\{N\}$ is a closed subset of $\mathbb{R}^3$, so $\mathbb{R}^3 \smallsetminus \{N\}$ is an open subset of $\mathbb{R}^3$ and thus

$$S^2 \smallsetminus \{N\} = (\mathbb{R}^3 \smallsetminus \{N\}) \cap S^2$$

is an open subset of S^2). So for every point of S^2, except the North pole, we have found an open neighborhood — $S^2 \smallsetminus \{N\}$ — homeomorphic to an open subset of $\mathbb{R}^2$ — $\mathbb{R}^2$ itself. It is then not hard to imagine that we can similarly define a stereographic projection from the South pole to show that also the North pole has a neighborhood homeomorphic to $\mathbb{R}^2$. So S^2 is a surface.

Another example we would like to describe is the surface of a donut. There are several ways to describe this surface, and we could for instance describe it as an explicit subset of $\mathbb{R}^3$, as for the sphere. We will take a different viewpoint, which will help us find many more examples of surfaces.

Start with a square. If we roll it up to glue the top side to the bottom side we get a cylinder. Now if we bend it to glue the right circle to the left circle, we get something that looks like the surface of a donut.

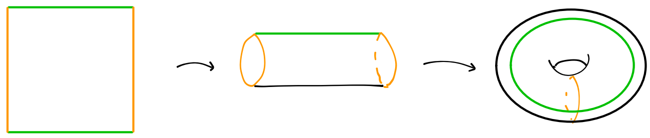

Figure 2: Folding and gluing a square to get the surface of a donut

To turn this procedure into a mathematical operation, we need to say what we mean by *gluing*. What we need is the concept of equivalence relation:

Definition 2.4. An *equivalence relation* $\sim$ on a set X is a relation which is:

- *reflexive*, i.e. $\forall x \in X$, $x \sim x$;
- *symmetric*, i.e. $\forall x, y \in X$, if $x \sim y$, then $y \sim x$;
- *transitive*, i.e. $\forall x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

If $x \sim y$, we say that x is *equivalent* to y.

Example 2.5. Let X be a collection of t-shirts. *Having the same color as* is an equivalence relation: clearly, any t-shirt has the same color as itself, so it's a reflexive relation. If t-shirt A has the same color as t-shirt B, then t-shirt B has the same color as t-shirt A, so the relation is symmetric. Finally, it is transitive: if t-shirt A has the same color as t-shirt B and t-shirt B has the same color as t-shirt C, then t-shirt A has the same color as t-shirt C.

We can also consider the relation *being smaller or equal than* (denoted $\leq$) on $\mathbb{R}$: this is not an equivalence relation, because it is not symmetric. For instance, $1 \leq 2$ but $2 \not\leq 1$.

If X is a set of words, we can define a relation as *having the same first letter or the same last letter as*. Say for instance

$$X = \{\text{car}, \text{cat}, \text{pear}\};$$

then the relation is not an equivalence relation either, because it's not transitive: *pear* is in relation with *car*, *car* is in relation with *cat*, but *pear* is not in relation with *cat*.

Exercise 2.6. Show that the relation *being homeomorphic to* on the set of topological spaces is an equivalence relation.

When we glue opposite sides of the square, what we actually want to do is to consider a point on the bottom side to be "the same as" the corresponding point on the top side, and similarly for right-hand side and left-hand side. To formalize this, we need to speak about *equivalence classes* and *quotienting* by an equivalence relation.

Definition 2.7. Let $\sim$ be an equivalence relation on a set X. The *equivalence class* $[x]$ of an element $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\}.$$

Exercise 2.8. If $\sim$ is an equivalence relation on a set X, then $x \sim y$ if and only if $[x] = [y]$.

Definition 2.9. The *quotient* of X by an equivalence relation $\sim$ is the set $X/_\sim$ of equivalence classes of elements of X:

$$X/_\sim = \{[x] \mid x \in X\}.$$

The *(natural) projection* from X to $X/_\sim$ is the map

$$\begin{aligned} \pi : X &\to X/\sim \\ x &\mapsto [x]. \end{aligned}$$

Note that the quotient is a subset of the set of subsets $\mathcal{P}(X)$ of X and that the projection map is surjective.

In Example 2.5, an equivalence class is the set of all t-shirts of a given color. If we put the t-shirts in piles, split by colors, the quotient space can be though as the set of piles.

Back to surfaces and topology: the idea is to define the surface of a donut as a quotient space of the square by the appropriate equivalence relation. The equivalence relation gives us the set we want to consider, but we also need a topology. In fact, given a topological space and an equivalence relation, there is a natural way of giving a topology to the quotient space, which is what we need:

Definition 2.10. Let (X, τ) be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/_\sim$ is given by

$$\{U \subset X/_\sim \mid \pi^{-1}(U) \in \tau\}.$$

Exercise 2.11. With respect to the quotient topology, the projection map is continuous.

Example 2.12. Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if $|x| = |y|$. It is not hard to see that it's an equivalence relation and that the equivalence class of $x \in \mathbb{R}$ is $[x] = \{x, -x\}$. So every equivalence class is a set of cardinality two, except for the equivalence class of 0, which is simply $\{0\}$. If we think of taking the quotient by $\sim$ as identifying each real number with its opposite, we can imagine it describes the operation of "folding" the real line along 0, so we can expect the quotient space to be homeomorphic to $[0, \infty[$. This is in fact correct, and here is how we can prove it: define the map

$$\begin{aligned} f : [0, \infty[&\to \mathbb{R}/_{\sim} \\ x &\mapsto [x]. \end{aligned}$$

First, note that it is a bijection:

- if $f(x) = f(y)$, then $[x] = [y]$, i.e. $x \sim y$, that is $|x| = |y|$. But as $x, y \geq 0$, $|x| = x$ and $|y| = y$, so we deduce $x = y$, proving that f is injective;
- let $[x] \in \mathbb{R}/_{\sim}$; then $[x] = f(|x|)$, so f is surjective.

To show that f is continuous, let U be an open set in $\mathbb{R}/_{\sim}$. Then by definition of the quotient topology, denoting by $\pi : \mathbb{R} \to \mathbb{R}/_{\sim}$ the projection map, we know that $\pi^{-1}(U)$ is open. Moreover note that, for any $x \in \mathbb{R}$,

$$\pi^{-1}([x]) = \{x, -x\}$$

and

$$f^{-1}([x]) = \{|x|\},$$

so

$$f^{-1}([x]) = \pi^{-1}([x]) \cap [0, \infty[.$$

Thus

$$f^{-1}(U) = \pi^{-1}(U) \cap [0, \infty[$$

which implies that the preimage of U is open.

Next we show that f is open, so let $U \subset [0, \infty[$ be an open set of $[0, \infty[$. Then one can show (exercise) that

$$\pi^{-1}(f(U)) = U \cup \{-x \mid x \in U\}$$

and that $U \cup \{-x \mid x \in U\}$ is open in $\mathbb{R}$, so that $f(U)$ is open, i.e. f is open.

So f is continuous, open and bijective, which means that it is a homeomorphism.

Another example is the following, which formalizes the fact that if we start from a segment and glue the two endpoints together we get a circle.

Example 2.13. Let $\sim$ be the equivalence relation on the interval $[0, 1]$ given by

- $t \sim t$ for every $t \in [0, 1]$,
- $0 \sim 1$,
- $1 \sim 0$.

Show that $[0, 1]/_{\sim}$ is homeomorphic to the circle S^1. *Hint: any* $(x, y) \in S^1$ *can be written as* $(\cos(2\pi t), \sin(2\pi t))$, *for* $t \in [0, 1[$. *Use this to define a map* $S^1 \to [0, 1]/_{\sim}$.

Let us look at the example we wanted, describing something like the surface of a donut (see Figure 3).

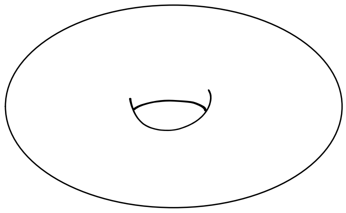

Figure 3: The surface of a donut

Example 2.14. Let $X = [0,1] \times [0,1]$ and $\sim$ the equivalence relation given by:

- $(x, y) \sim (x, y)$ for every $(x, y) \in X$,
- $(x_1, y_1) \sim (x_2, y_2)$ if $\{y_1, y_2\} = \{0, 1\}$ and $x_1 = x_2$,
- $(x_1, y_1) \sim (x_2, y_2)$ if $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$,
- if $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, then $(x_1, y_1) \sim (x_3, y_3)$.

The second condition is the one describing the gluing of top side to bottom side and the third describes the gluing of right-hand side to left-hand side. Note that a point which is not on a side of the square is in relation only with itself, i.e. π is injective on $]0,1[\times]0,1[$.

Let $T = X/_\sim$ (with the quotient topology). We have described the gluing operation mathematically now; let us check that T is a surface.

Let $P = [(x_0, y_0)] \in T$; we distinguish three cases.

Case 1: (x_0, y_0) is not on a side of the square. Then there is some small $\varepsilon > 0$ such that $B_\varepsilon((x_0, y_0)) \subset]0,1[\times]0,1[$. We claim that $U_P := \pi(B_\varepsilon((x_0, y_0)))$ is an open neighborhood of $[(x_0, y_0)]$ homeomorphic to $B_\varepsilon((x_0, y_0))$: indeed, since π is injective on $]0,1[\times]0,1[$

- $\pi^{-1}(U_P) = B_\varepsilon((x_0, y_0))$ which is open, so U_P is open;
- $\pi : B_\varepsilon((x_0, y_0)) \to U_P$ is a homeomorphism (exercise).

Case 2: (x_0, y_0) belongs to a side, but not to a corner. For simplicity, suppose $x_0 = 0$. Then there is some $\varepsilon > 0$ such that $B_\varepsilon((0, y_0))$ does not contain any corner. We define a neighborhood U_P by

$$U_P = \{[(x, y)] \mid (x, y) \in B_\varepsilon((0, y_0)), x \geq 0\} \cup \{[(x, y)] \mid (x, y) \in B_\varepsilon((1, y_0)), x \leq 1\}.$$

As before, the claim is that U_P is an open neighborhood of P and it's homeomorphic to $B_\varepsilon((0, y_0))$. The idea is that it's made up by two half disks, glued together along a diameter.

Case 3: (x_0, y_0) is a corner, so $P = [(0, 0)]$ (note that all four corners are equivalent to each other). Then the neighborhood is obtained as the union of four quarter disks, as in Figure 4. A good exercise is to make this precise.

The surface obtained this way is called *torus*.

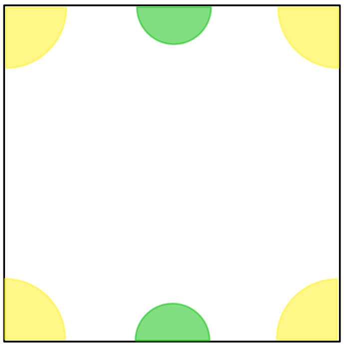

Figure 4: The neighborhoods

If we think about the way we proved that the space in Example 2.14 is a surface, we may realize that we have found a way to describe many surfaces: we start with a polygon and we glue sides, two by two. What matters is that there is no side left alone, because we otherwise we get points with no open neighborhood homeomorphic to a disk (we get only half disks!). Alternatively we start with a polygon where some sides are included and some aren't; then we need to glue together only the sides that are included. Here is an example.

Example 2.15. Let $X = [0,1]\times]0,1[$ and $\sim$ the equivalence relation given by $(x_1, y_1) \sim (x_2, y_2)$ if either:

- $(x_1, y_1) = (x_2, y_2)$, or
- $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = 1 - y_2$.

What is happening is that we are taking a square and gluing right-hand side to left-hand side *with a twist*. We can do it with a piece of paper. Try and notice how the object you get is one-sided: without passing by the border (nor going through the surface), we can go from what seems to be one side of the surface to the opposite! Instead for the torus there are two distinct sides, and we cannot go from one to the other without going through the surface.

As for the torus, we can show that the space $M = X/_\sim$ is a surface, which is called *Möbius band* (or *Möbius strip*).

3 Lecture 3

In the last class we realized that if we consider a set of polygons, where some sides are included and some aren't, and we glue the sides that are included in pairs, we obtain a surface. Writing down the equivalence relation corresponding to gluing sides can be complicated, but we can describe it in a more pictorial way, as follows.

We draw the polygons in the plane; if two sides are to be glued together, we label them with the same letter. But this is not enough: we also need to know how to glue the two edges. The convention will be as follows: we add arrows to the sides, and choose the unique gluing so that the starting point of one side is glued to the starting point of the other side, the same for the end points, and for points in between we glue proportionally to the distance, i.e. if the two segments are of the form $[p, q]$ and $[r, s]$, the first has length ℓ_1 and the second ℓ_2, and the arrows point from p to q and from r to s, the point $x \in [p, q]$ at distance $t\ell_1$ (for some $t \in [0, 1]$ from p is glued to the point $y \in [r, s]$ at distance $t\ell_2$ from r. Essentially, we glue so that the orientations of the two sides match.

See Figure 5 for how we do this for the torus and the Möbius band.

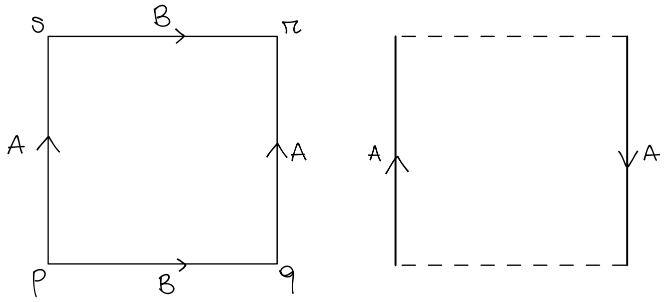

Figure 5: Side identifications for the torus and the Möbius band

Figure 6 gives another example of gluing. We know that the quotient space is a surface, and with some effort we can see that it looks like the surface in Figure 7. But in general, given a polygon with some gluings, it is pretty hard to visualize the final surface. Luckily, there are better ways than starting with a piece of fabric and try to figure out what happens. The point is that there is a classification of surfaces up to homeomorphisms and there are topological invariants (properties which are invariant up to homeomorphism) which can be computed by looking at a presentation of a surface as polygons with sides identified and allow us to deduce which surface we are dealing with. Our next task is to define these invariants and state the classification.

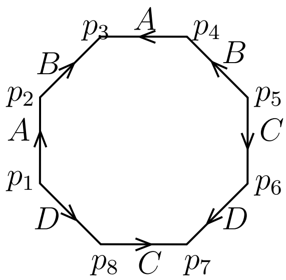

Figure 6: An octagon with sides identified; the p_i are the vertices

3.1 Compactness

Definition 3.1. A subset Y of a topological space (X, τ) is *compact* if for every collection $\{U_i \mid i \in I\}$ of open sets covering Y (i.e. such that $\bigcup_{i \in I} U_i \supset Y$), there is a finite subcollection which covers Y.

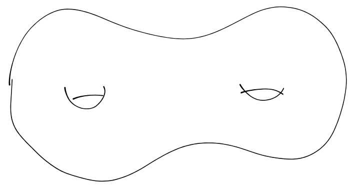

Figure 7: A surface with two “holes”

Example 3.2. $\mathbb{R}$ is not compact: consider the open cover $\{U_n =]-n, n[\mid n \in \mathbb{N}\}$. If $F \subset \mathbb{N}$ is finite, it has a maximum m, so

$$\bigcup_{n\in F}]-n, n[=]-m, m[\neq \mathbb{R}.$$

On the other hand, in $\mathbb{R}^n$ we have an easy way to describe compact subsets: by the Heine–Borel theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (where a subset in $\mathbb{R}^n$ is bounded if it is contained in some ball around the origin).

There is a useful fact to keep in mind:

Proposition 3.3. *The continuous image of a compact space is compact (i.e. if (X, τ) and (Y, σ) are topological spaces, X is compact and $f : X \to Y$ is a continuous function, then $f(X)$ is compact).*

In particular we have the following useful consequence:

Corollary 3.4. *The quotient of a compact topological space via an equivalence relation is compact.*

Proof. If (X, τ) is compact and $\sim$ is an equivalence relation, the projection map $\pi : X \to X/_\sim$ is a continuous surjective map. So $X/_\sim$ is the continuous image of a compact set and hence it is compact. □

Example 3.5. The torus T is the quotient of $[0, 1] \times [0, 1]$, which is a compact space, so the torus is compact. On the other hand, the Möbius band M is the quotient of $[0, 1]\times]0, 1[$, which is not compact, so we cannot apply the corollary. In fact, the Möbius band is not compact: for any integer $n \geq 3$, we can define

$$U_n = \pi\left([0, 1] \times \left]\frac{1}{n}, 1 - \frac{1}{n}\right[\right),$$

where $\pi : [0, 1]\times]0, 1[\to M$ is the natural projection. It is an exercise to check that the U_n are open sets in M and

$$\bigcup_{n\geq 3} U_n = M.$$

On the other hand, for any finite $F \subset \{n \in \mathbb{N} \mid n \geq 3\}$,

$$\bigcup_{n\in F} U_n = U_N \neq M,$$

where $N = \max_{n \in F} n$. So we found an open cover of M containinig no finite subcollection covering M.

Remark 3.6. It is not true that the quotient of a noncompact space is noncompact: for instance, one can check that $([0,1]\times]0,1[)/_\sim$, where $\sim$ is the equivalence relation given by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$, is compact (and in fact homeomorphic to S^1).

3.2 Orientability

A funny surface we have seen is the Möbius band. Some surfaces will contain it as a subspace and others won't. We want to make a distinction between these two cases:

Definition 3.7. A surface S is *orientable* if it contains no Möbius band, i.e. if there is no homeomorphism between a Möbius band and a subspace of S. A surface is *non-orientable* otherwise.

The intuition is that orientable surfaces "have two sides", while non-orientable ones don't. So for instance the sphere and the torus are orientable surfaces, while the one – usually called *Klein bottle* – obtained by a square with identifications as in Figure 8 isn't. In the same figure we have highlighted a Möbius band contained in the Klein bottle.

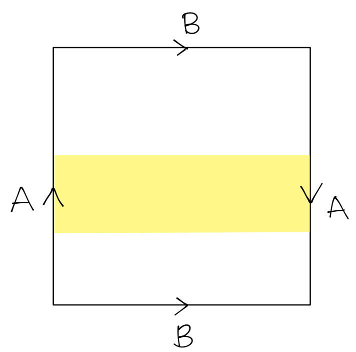

Figure 8: The side identifications for the Klein bottle

Another useful fact is that orientable surfaces can be embedded in $\mathbb{R}^3$, while compact non-orientable ones cannot. In Figure 9 you can see what were to happen if we tried to embed the Klein bottle in space: we get some self-intersection which is not supposed to be there.

In practice, we will consider only compact orientable surfaces, which can be nicely drawn in $\mathbb{R}^3$ and are easy to classify. One last concept that we need, before stating the classification, is that of *path-connectedness*: this will describe mathematically the fact that the surfaces we consider are made of "one piece".

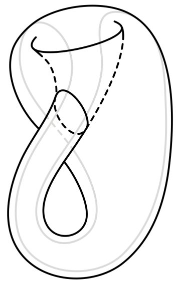

Figure 9: Visualizing the Klein bottle (from Wikipedia): the circle is the self-intersection we obtain when trying to embed it in $\mathbb{R}^3$

3.3 Path-connectedness

Definition 3.8. Let (X, τ) be a topological space. A *path* from $x \in X$ to $y \in X$ is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

We can think of paths as wiggly lines drawn on a topological space. Generic paths can self-intersect as much as they want, and actually they can even be completely degenerate: for any topological space (X, τ) and any point $x \in X$, the map $f : [0, 1] \to X$ given by $f(t) = x$ for every $t \in [0, 1]$ is a path from x to x.

Definition 3.9. X is *path-connected* if for every pair of distinct points $x \neq y \in X$ there is a path from x to y.

Intuitively, a space is path-connected if we can walk on it from any point to any other point. Here are some examples:

Example 3.10. 1. $\mathbb{R}$ is path-connected: given $x, y \in \mathbb{R}$,

$$\begin{aligned} f : [0, 1] &\to \mathbb{R} \\ t &\mapsto x + t(y - x) \end{aligned}$$

is a path between them.

2. S^1 is path-connected: if $p, q \in S^1$, then we can write $p = (\cos(\theta_1), \sin(\theta_1))$ and $q = (\cos(\theta_2), \sin(\theta_2))$, for some $\theta_1, \theta_2 \in [0, 2\pi[$. Then a path between p and q is given by

$$\begin{aligned} f : [0, 1] &\to S^1 \\ t &\mapsto (\cos(\theta_1 + t(\theta_2 - \theta_1)), \sin(\theta_1 + t(\theta_2 - \theta_1))). \end{aligned}$$

Note that we could not consider as a path between them a straight line in $\mathbb{R}^2$: the path needs to be contained in S^1!

3. $X = [0, 1] \cup [2, 3]$ is not path-connected: if $f : [0, 1] \to X \subset \mathbb{R}$ is a continuous function with $f(0) = 1$ and $f(1) = 2$, by the intermediate value theorem f would need to take every value between $f(0) = 1$ and $f(1) = 2$. So f is not a function with values in X, that is, we cannot walk from 1 to 2 while staying in X.

3.4 Classification

We can now state the classification of surfaces:

Theorem 3.11. *Let S be a path-connected, compact, orientable surface. Then there is a unique integer $g \geq 0$ such that S is homeomorphic to S_g, where $S_0 = S^2$ and for every $g \geq 1$ S_g is the surface obtained by a $4g$-gon with the identifications in Figure 10.*

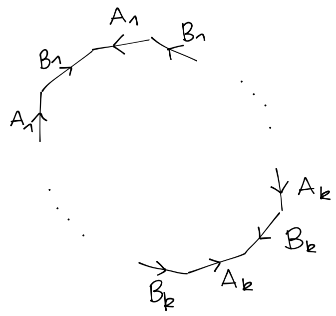

Figure 10: Side identifications on a $4g$-gon

We can also visualize these surfaces better, as in Figure 11. The integer g is what is called *genus* of a surface, so that S_g is said to be a *surface of genus g*. Informally, the genus is the number of "holes", so that a surface of genus g can be obtained by attaching g handles to a sphere. In particular S_0 is the sphere and S_1 is the torus.

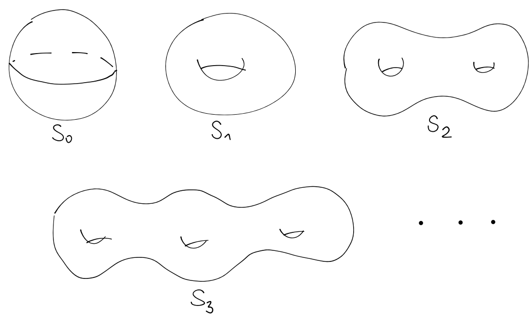

Figure 11: The surfaces S_g

We describe now how to figure out if a surface, given as polygons with side identifications, is path-connected, orientable and compact and how we can compute a topological invariant, called Euler characteristic, which will allow us to compute the genus.

3.5 Checking compactness

We will use the fact that the quotient of a compact space is compact. Moreover, it is not hard to show that a disjoint union of finitely many compact spaces is compact. Thus:

Proposition 3.12. *Let S be a surface obtained as quotient of a finite collection of compact polygons, with sides identified in pairs. Then S is compact.*

3.6 Checking path-connectedness

Note first that if we start from a single polygon P with sides identified in pairs, the surface S we obtain is path-connected: indeed, given two points $x, y \in S$, and denoting by $\pi \colon P \to S$ the natural projection, we can choose $p \in \pi^{-1}(x)$ and $q \in \pi^{-1}(y)$ and find a path f between p and q in the polygon. Then $\pi \circ f$ is a path between x and y. If we have multiple polygons, we just need to guarantee that we can go from every polygon to every other polygon via some sequence of side identifications.

Proposition 3.13. *Let S be a surface obtained as quotient of a finite collection of polygons $P_i, i = 1, \ldots, k$. Then S is path-connected if and only if for any two indices $i \neq j \in \{1, \ldots, k\}$ there is a sequence $i_1 = i, \ldots, i_m = j$ of indices such that P_{i_l} has a side identified with a side of $P_{i_{l+1}}$, for every $l = 1, \ldots, j-1$.*

Example 3.14. The surface on the left-hand side of Figure 12 is not path connected (there is no way of having a path from p to q, because no side of the first polygon is identified with a side of the second polygon. On the other hand, the surface on the right-hand side is path connected: for instance, for going from r to s we can follow the path drawn in green, which corresponds crosses the polygons P_2, P_1, P_3 in this order, through the common sides A and C.

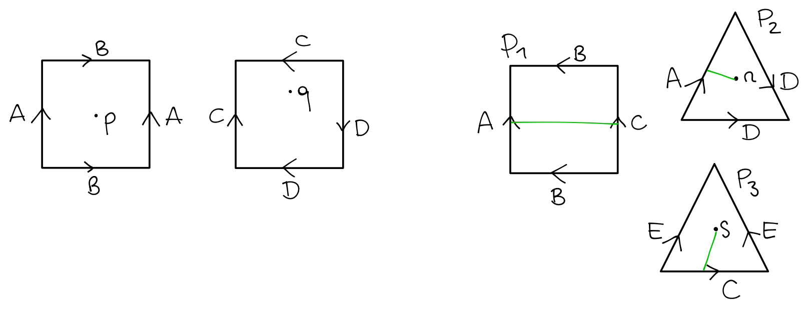

Figure 12: Two surfaces

3.7 Checking orientability

For orientability, we give a criterion only in the case in which we obtain it as the quotient of a single polygon.

Proposition 3.15. *Let S be a surface obtained by identifying in pairs the sides of a polygon P. Then S is orientable if and only if the following holds:*

for any pair of sides which are identified, the (counterclockwise) orientation of the boundary agrees with orientation given by the arrow for exactly one of the two sides.

One direction is easy to show: if we have two sides identified whose arrows have both the same or both the opposite orientation as the counter-clockwise orientation, we can find an embedded Möbius band as in Figure 13. It is harder to show that the orientation condition is also sufficient for orientability.

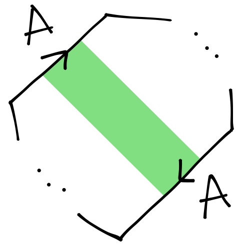

Figure 13: Finding a Möbius band

3.8 Computing genus and Euler characteristic

Instead of directly compute the genus of a compacted, path-connected, orientable surface, we will show how to computen another invariant, called *Euler characteristic*, and show how this can be used to compute the genus. We have:

Theorem 3.16. *Let S be a surface. Then we can associate to S a number $\chi(S)$, called Euler characteristic, which is invariant under homeomorphism (i.e. if two surfaces are homeomorphic, they have the same Euler characteristic) and satisfies*

$$\chi(S_g) = 2 - 2g.$$

If S is a surface obtained by finitely many polygons with sides identified in pairs, v denotes the number of equivalence classes of vertices of polygons on the surface, e half the number of edges of the polygons and f the number of polygons. Then

$$\chi(S) = v - e + f.$$

So now if we start from a compact, path-connected and orientable surface S obtained by polygons with side identifications, we know by the classification theorem that there is $g \geq 0$ so that S is homeomorphic to S_g. This implies, by Theorem 3.16, that $\chi(S) = \chi(S_g) = 2 - 2g$, so we can find g by computing the Euler characteristic of S. Then

$$g = 1 - \frac{\chi(S)}{2}.$$

Example 3.17. Let's first look at the torus T, obtained as a square with identifications as in Figure 5. Since the two sides labelled with A are identified, the vertices p and q project to the same point on the torus. But also the sides labelled with B are identified, so p and s are identified, and so are q and r. So all four vertices project to a single point on the torus, i.e. $v = 1$. The square has four edges, so $e = 2$, and we start with a single polygon, so $f = 1$. So

$$\chi(T) = 1 - 2 + 1 = 0,$$

which means that the genus of the torus is

$$g = 1 - \frac{0}{2} = 1.$$

Example 3.18. Let us look at the surface S given by the polygon with identifications as in Figure 14.

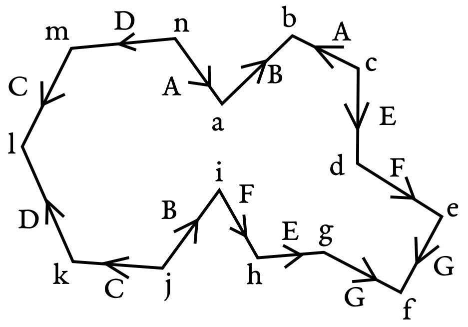

Figure 14: A surface

Our goal is to use the classification theorem to find out which surface S is. We know that:

- S is the quotient of a single compact polygon, so it is compact and path-connected by Propositions 3.12 and 3.13;
- if we follow the boundary of the octagon going in the counterclockwise direction, the arrows of any pair of sides with the same label come with different orientations, so S is orientable by Proposition 3.15.

So we know, by the classification theorem, that S is homeomorphic to S_g, for some $g \geq 0$. So we just need to compute g, and for compact, orientable, path-connected surfaces the genus can be deduced by the Euler characteristic. For S, we have $f = 1$ and $e = \frac{14}{2} = 7$. Moreover, $a \sim b$, because they are both the starting point of a side labelled A; $b \sim i$, because they are both the endpoint of a side labelled B; $i \sim d$, by looking at F... repeating this argument we can show that all vertices are equivalent, except for f. So there are two equivalence classes, $[a]$ and $[f]$, i.e. $v = 2$. Thus

$$\chi(S) = 2 - 7 + 1 = -4$$

which means that the genus of S is

$$g = 1 - \frac{-4}{2} = 3,$$

i.e. S is homeomorphic to S_3.

Exercise 3.19. Verify that the surface in Figure 15 is homeomorphic to S_2.

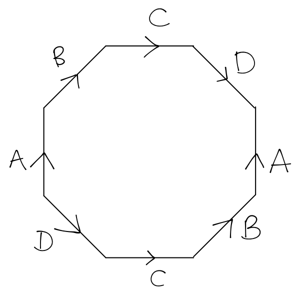

Figure 15: An octagon with opposite sides identified

3.9 Simplicial complexes and triangulations

In Section 3.8 we have mentioned that we can associate a topological invariant to a surface, the Euler characteristic. Actually, this is an invariant which can be associated to much more general topological spaces, including for instance manifolds of every dimension. It can be defined in particular for a class of topological spaces called simplicial complexes, which we define here.

Simpicial complexes are obtained by gluing together simple building blocks, called *simplices*.

Definition 3.20. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. The *convex hull* of $v_1, \ldots, v_k$ is the set

$$\mathrm{Conv}(v_1, \ldots, v_k) := \left\{ \sum_{i=1}^{k} t_i v_i \mid t_i \in [0,1], \sum_{i=1}^{k} t_i = 1 \right\}.$$

The *standard k-simplex* Δ^k is the convex hull of $e_0, \ldots, e_k \in \mathbb{R}^k$, where $e_0 = (0, \ldots, 0)$ and for every $i \in \{1, \ldots, n\}$, e_i is the vector whose coefficients are all zero, except for the i-th coefficient, which is one. A *j-face* of Δ^k (for $j \leq k$) is the convex hull of $j+1$ points among $e_0, \ldots, e_k$.

So the standard 0-simplex is a point, the standard 1-simplex is a segment, the standard 2-simplex is a triangle and the standard 3-simplex is a tetrahedron (see Figure 16). The 0-faces of 3-simplex are its vertices, its 1-faces its edges and its 2-faces its triangular faces.

Next time we will use these building blocks to construct topological spaces.

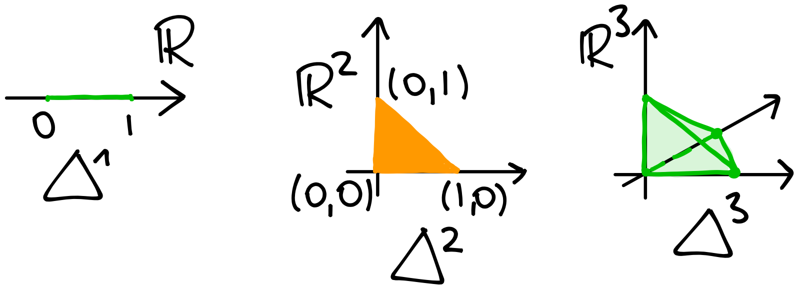

Figure 16: The standard 1-, 2-, and 3-simplex

4 Lecture 4

As mentioned last time, the idea of a simplicial complex is to take a bunch of simplices and glue them together along their faces. For instance, let us look at the space in Figure 17:

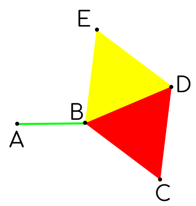

Figure 17: A topological space

We cant think of it as given by gluing a 1-simplex, the segment of vertices A and B, and two 2-simplices, the triangles of vertices B, C, D and B, D, E. To encode which simplices we are considering, we can look at which vertices form simplices. So here we have the vertices, A, B, C, D, E, the 1-simplices AB, BC, CD, BD, BE, ED and the 2-simplices BCD and BDE. So we can associate to this gluing of simplices the collection of sets:

$$\{\{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{A, B\}, \{B, C\}, \{C, D\},$$
$$\{B, D\}, \{B, E\}, \{E, D\}, \{B, C, D\}, \{B, D, E\}\}.$$

This is what is called the the abstract simplicial complex defining the space.

4.0.1 Abstract simplicial complexes

Definition 4.1. A *(finite) abstract simplicial complex* K is given by a set $V(K)$, whose lements are called *vertices*, and a set $S(K)$ of non-empty subsets of $V(K)$ called *abstract simplices*) such that

1. if $\sigma \in S(K)$ and $\tau \subset \sigma$, then $\tau \in S(K)$;
2. for every $v \in V(K)$, $\{v\} \in S(K)$.

We denote by $s_i(K)$ the number of sets of $S(K)$ of cardinality i.

Example 4.2. Suppose $V(K) = \{a, b, c\}$.

- $S(K) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ defines an abstract simplicial complex.
- $S(K) = \{\{a\}, \{b\}, \{c\}, \{a, b, c\}\}$ doesn't define an abstract simplicial complex, because there are subsets of $\sigma = \{a, b, c\}$ which don't belong to $S(K)$.
- $S(K) = \{\{a\}, \{b\}\}$ doesn't define an abstract simplicial complex, because $\{c\} \notin S(K)$.

4.0.2 (Topological) simplicial complexes

Now the idea is to start with an abstract simplicial complex K, replace a set of size $k + 1$ by a k-simplex and glue the simplices together according to the subset relation. For instance, for K_1 from Example 4.12, we take three points A, B, C, one each for $\{a\}, \{b\}$ and $\{c\}$, then two segments I, J, one for $\{a, b\}$ and one for $\{a, c\}$. Identify one vertex of I with A and one with B, and one vertex of J with A and one with C. So in practice we get two segments glued together along one of their vertices (see Figure 18).

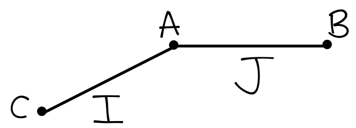

Figure 18: The geometric realization of K_1 from Example 4.12

To formalize this construction we do the following:

Definition 4.3. Let $\sigma = \{a_0, \ldots, a_k\}$ be a finite set. The *simplex realization of* σ is the topological space $|\sigma|$ whose underlying set is the set of formal linear combinations

$$\left\{ \sum_{i=0}^{k} t_i a_i \mid t_i \in [0, 1], \sum_{i=0}^{k} t_i = 1 \right\}$$

and the topology is induced by identifying this set with the standard k-simplex (i.e. we consider the bijection

$$\theta : \left\{ \sum_{i=0}^{k} t_i a_i \mid t_i \in [0, 1], \sum_{i=0}^{k} t_i = 1 \right\} \to \Delta^k$$

$$\sum_{i=0}^{k} t_i a_i \mapsto \sum_{i=0}^{k} t_i e_i$$

and declare that the open sets in $\left\{ \sum_{i=0}^{k} t_i a_i \mid t_i \in [0, 1], \sum_{i=0}^{k} t_i = 1 \right\}$ to be the preimages via θ of the open sets in Δ^k.

This first definition formalizes the association of a k-simplex to a set of cardinality $k+1$. Next we need to glue the simplices according to the combinatorial pattern given by an abstract simplicial complex.

Definition 4.4. Let K be an abstract simplicial complex. The *topological realization* $|K|$ of K is the topological space given by

$$|K| = \left(\bigsqcup_{\sigma \in S(K)} |\sigma| \right) /_{\sim}$$

where $\sim$ is the equivalence relation so that $p \sim q$ if they are the same formal linear combination.

Let us see how the definition works in the example we saw before.

Example 4.5. Let $V(K) = \{a, b, c\}$ and $S(K) = \{\{a\}, \{b\}, \{c\}, \sigma = \{a, b\}, \tau = \{a, c\}\}$. Then

$$\begin{aligned} |\sigma| &= \{ta + (1-t)b \mid t \in [0,1]\} \\ |\tau| &= \{ta + 1 - t)c \mid t \in [0,1]\} \\ |\{a\}| &= \{a\} \\ |\{b\}| &= \{b\} \\ |\{c\}| &= \{c\} \end{aligned}$$

and the equivalence relation identifies

1. $a \in |\{a\}|$, one vertex of $|\sigma|$ and one vertex of $|\tau|$
2. $b \in |\{b\}|$ and the other vertex of $|\sigma|$
3. $c \in |\{c\}|$ and the other vertex of $|\tau|$.

So indeed $|K|$ is given by two segments glued together along a vertex.

Exercise 4.6. Draw the topological realization of the abstract simplicial complex K given by

- $V(K) = \{a, b, c, d\}$
- $S(K) = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, d\}\}$

Remark 4.7. In any simplicial complex realization, each simplex realization has distinct vertices (two vertices of a simplex are never identified with each other). Moreover, if σ, τ are abstract simplices in $S(K)$, $|\sigma| \cap |\tau| = |\sigma \cap \tau|$, i.e. if the intersection of two simplex realizations is not empty, it is a single face of both of them.

Topological realizations of abstract simplicial complexes are the basic examples of simplicial complexes:

Definition 4.8. A *(finite) simplicial complex* is a triple (X, K, f), where X is a topological space, K is an abstract simplicial complex and $f : |K| \to X$ is a homeomorphism. We then say that X is *triangulable* and we call (K, f) a *triangulation* of X.

Example 4.9. The circle S^1 is triangulable, and a triangulation is given by (K_1, f), where f is the homeomorphism informally described in Figure 19

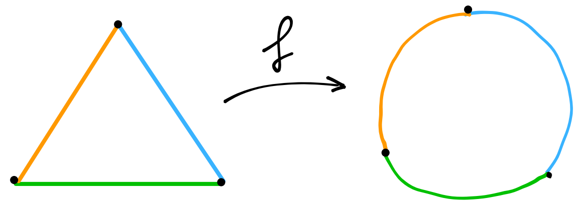

Figure 19: The homeomorphism f restricts to three homeomorphisms, one from each side of the triangle to the arc on the circle of the same color.

It turns out that we can define the Euler characteristic of any triangulable space.

Proposition 4.10. *Let X be a triangulable space and K, f a triangulation of X. There is a topological invariant, called* Euler characteristic *of X, which can be computed as*

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i s_i(K),$$

where

$$s_i(K) = |\{\sigma \in S(K) \mid \sigma \text{ has cardinality } i+1\}|.$$

Note that since we are considering finite simplicial complexes the sum in the proposition is finite ($s_i(K) = 0$ for all i large enough). Also, part of the content of the proposition is that different triangulations of the same topological space will give the same Euler characteristic, which is far from obvious.

In particular, we can also use triangulation of surfaces to compute their Euler characteristic. One disadvantage of this approach is that in general a triangulation of a surface contains many simplices. For instance, we can look at the case of the torus. In Figure 20 we see two decompositions of the torus in pieces that look like triangles, but the first one isn't a triangulation: first of all, the two pieces are not homeomorphic to simplex realizations in a simplicial complex, because the three "vertices" are a single point on the surface. But even if this were not the case, there is another problem: the intersection of the two pieces is the union of three "sides", and we saw before that the intersection of two simplex realization, if not empty, is a single face of either simplex ralization. The second one is a triangulation, but it contains 18 triangles.

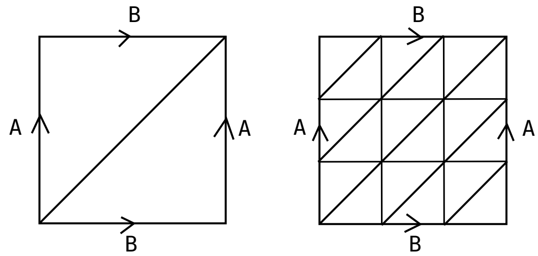

Figure 20: Two decompositions of the torus into triangles

4.1 Simplicial maps

We have defined simplicial complexes; the standard maps to consider between them are those which preserve the decomposition in simplices. These maps will be called simplicial maps. We will define them first at the level of abstract simplicial complexes and then at the level of the associated topological spaces.

Definition 4.11. A *simplicial map* between two abstract simplicial complexes K_1 and K_2 is a map $f : V(K_1) \to V(K_2)$ such that if $\sigma \in S(K_1)$, then $f(\sigma) \in S(K_2)$.

Example 4.12. Consider the abstract simplicial complexes K_1, K_2 and K_3 given by

$$\begin{aligned}
V(K_1) &= \{a, b, c\} \\
S(K_1) &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\} \\
V(K_2) &= \{x, y, z, t\} \\
S(K_2) &= \{\{x\}, \{y\}, \{z\}, \{t\}, \{x, y\}, \{x, z\}, \{y, z\}\} \\
V(K_3) &= \{\alpha, \beta, \gamma\} \\
S(K_3) &= \{\{\alpha\}, \{\beta\}, \{\gamma\}\}
\end{aligned}$$

The map

$$\begin{gathered}
f_1 : V(K_1) \to V(K_2) \\
a \mapsto x \\
b \mapsto z \\
c \mapsto y
\end{gathered}$$

defines a simplicial map from K_1 to K_2, because

$$f_1(\{a, b\}) = \{x, z\} \in S(K_2)$$

and

$$f_1(\{a, c\}) = \{x, y\} \in S(K_2).$$

On the other hand, the map

$$\begin{gathered}
f_2 : V(K_1) \to V(K_2) \\
a \mapsto z \\
b \mapsto y \\
c \mapsto t
\end{gathered}$$

does not define a simplicial map from K_1 to K_2, because

$$f_2(\{a, c\}) = \{z, t\} \notin S(K_2).$$

On the other hand, the only way to define a simplicial map from K_1 to K_3 is to map all vertices of K_1 to a single vertex of K_3, otherwise there will be a simplex of K_1 which is sent to something that's not a simplex of K_3.

Definition 4.13. A *simplicial isomorphism* between two abstract simplicial complexes K_1 and K_2 is a simplicial map f which is a bijection between $V(K_1)$ and $V(K_2)$ and induces a bijection between $S(K_1)$ and $S(K_2)$. Two abstract simplicial complexes are *combinatorially equivalent* if there is a simplicial isomorphism between them.

Note that a simplicial map f is a simplicial isomorphism if and only if there is an inverse simplicial map, i.e. a simplicial map g such that $g = f^{-1}$.

Moreover, in practice two abstract simplicial complexes are combinatorially equivalent if the only difference is that we are calling the vertices with different names.

Example 4.14. Let K_1 be as in Example 4.12 and K_4 be given by $V(K_4) = \{x, y, z\}$ and

$$S(K_4) = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}\}.$$

Then K_1 and K_2 are combinatorially equivalent, and a simplicial isomorphism between them is given by

$$\begin{aligned} f : V(K_1) &\to V(K_4) \\ a &\mapsto y \\ b &\mapsto x \\ c &\mapsto z. \end{aligned}$$

Note that any simplicial isomorphism between them needs to send a to y, because these are the only points belonging to two simplices of cardinality 2.

Note that it is not enough for a simplicial map to be a bijection of the set of vertices to be a simplicial isomorphism, as the following example shows.

Example 4.15. Let K_1 be as in Example 4.12 and K_5 be given by $V(K_5) = \{x, y, z\}$ and

$$S(K_5) = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, z\}\}.$$

Then any bijection $V(K_1) \to V(K_5)$ defines a simplicial map, but neither is a simplicial isomorphism, because K_5 has one simplex more than K_1.

We now define the maps between topological realizations of simplicial complexes:

Definition 4.16. Let f be a simplicial map from K_1 to K_2, where K_1, K_2 are abstract simplicial complexes. We define the map

$$|f| : |K_1| \to |K_2|$$

given by

$$\sum_{i=0}^{k} t_i a_i \mapsto \sum_{i=0}^{k} t_i f(a_i).$$

Essentially, the map f tells us which vertices are sent to which vertices and then we extend the map in the most obvious way to the topological realizations of the simplices. Here we see why it's important that, in the definition of simplicial map, simplices are sent to simplices: otherwise, we would not be able to extend the map to the topological realizations of the simplices.

It is not so difficult to show that:

Lemma 4.17. *Let f be a simplicial map from K_1 to K_2, where K_1, K_2 are abstract simplicial complexes. Then $|f|$ is continuous.*

Example 4.18. Suppose K_1 and K_2 are the abstract simplicial complexes given by

$$\begin{aligned} V(K_1) &= \{x, y\} \\ S(K_1) &= \{\{x\}, \{y\}, \{x, y\}\} \\ V(K_2) &= \{a, b, c\} \\ S(K_2) &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\} \end{aligned}$$

We can consider the simplicial maps f and g given by

$$f(x) = a, f(y) = b$$

and

$$g(x) = g(y) = b.$$

Then the map $|f|$ will send the segment $|K_1|$ homeomorphically to the segment $|\{a, b\}| \subset |K_2|$, while the map $|g|$ is the map sending every point of $|K_1|$ to one of the vertices of $|K_2|$ (the vertex b).

We end with the following fact:

Lemma 4.19. *If f is a simplicial isomorphism, $|f|$ is a homeomorphism.*

The idea of the proof is to show that $|f^{-1}|$ is a continuous inverse of $|f|$.