Homework

Basics in Mathematics – Geometry

Exercise 1. Show that the collection

$$\tau = \{U \subset \mathbb{R} \mid \mathbb{R} \smallsetminus U \text{ is either finite or } \mathbb{R}\}$$

is a topology on $\mathbb{R}$.

Solution. We need to check the three conditions that a collection $\tau \subset \mathcal{P}(\mathbb{R})$ needs to satisfy to be a topology:

1. since $\mathbb{R} \smallsetminus \emptyset = \mathbb{R}$, $\emptyset \in \tau$; moreover, $\mathbb{R} \smallsetminus \mathbb{R} = \emptyset$, which is finite, thus $\mathbb{R} \in \tau$. So the first condition is satisfied;
2. let $\{U_i \mid i \in I\}$ a collection of sets $U_i \in \tau$. Then:

   $$\mathbb{R} \smallsetminus \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{R} \smallsetminus U_i)$$

   and we have two possibilities:

   - either $\mathbb{R} \smallsetminus U_i = \mathbb{R}$ for every $i \in I$, in which case

     $$\mathbb{R} \smallsetminus \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{R} \smallsetminus U_i) = \mathbb{R},$$

   - or there is $i_0 \in I$ such that $\mathbb{R} \smallsetminus U_i$ is finite, in which case

     $$\mathbb{R} \smallsetminus \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{R} \smallsetminus U_i) \subset \mathbb{R} \smallsetminus U_{i_0},$$

     and thus $\mathbb{R} \smallsetminus \left(\bigcup_{i \in I} U_i\right)$ is finite,

   in both cases, we deduce that $\bigcup_{i \in I} U_i \in \tau$, as required;
3. let $U_1, \ldots, U_n$ a finite collection of sets $U_i \in \tau$; then

   $$\mathbb{R} \smallsetminus \left(\bigcap_{i=1}^{n} U_i\right) = \bigcup_{i=1}^{n} (\mathbb{R} \smallsetminus U_i)$$

   and as before we can distinguish two situations:

   - either $\mathbb{R} \smallsetminus U_i$ is finite for every i, in which case

     $$\mathbb{R} \smallsetminus \left(\bigcap_{i=1}^{n} U_i\right) = \bigcup_{i=1}^{n} (\mathbb{R} \smallsetminus U_i)$$

     is finite,

- or there is $i_0 \in I$ such that $\mathbb{R} \setminus U_i = \mathbb{R}$, in which case

$$\mathbb{R} \setminus \left(\bigcap_{i=1}^{n} U_i\right) = \bigcup_{i=1}^{n} (\mathbb{R} \setminus U_i) = \mathbb{R};$$

in both cases, we deduce that $\bigcap_{i=1}^{n} U_i \in \tau$, as required.

Exercise 2. Let (X, τ) be a topological space and Y a subset, with the subspace topology. Show that the inclusion map

$$\begin{aligned} \iota : Y &\hookrightarrow X \\ y &\mapsto y \end{aligned}$$

is continuous.

Solution. To prove that ι is continuous we need to show that the preimage of any $U \in \tau$ is open in Y. Since

$$\iota^{-1}(U) = \{y \in Y \mid \iota(y) \in U\} = \{y \in Y \mid y \in U\} = Y \cap U,$$

$\iota^{-1}(U)$ is open in Y by definition of the subspace topology.

Exercise 3. Let (X, τ) and Y, σ be the topological spaces given by $X = \{a, b, c\}$, $\tau = \mathcal{P}(X)$, $Y = \{x, y, z\}$, $\sigma = \{\emptyset, \{x\}, Y\}$. Consider the map

$$\begin{aligned} f : X &\to Y \\ a &\mapsto x \\ b &\mapsto y \\ c &\mapsto z. \end{aligned}$$

Show that f is continuous. Is f a homeomorphism? If yes, prove it. If not, justify why.

Solution. Let us show first that f is continuous, i.e. that for every $U \in \sigma$, $f^{-1}(U) \in \tau$. We have:

$$\begin{aligned} f^{-1}(\emptyset) &= \emptyset \in \tau \\ f^{-1}(\{x\}) &= \{a\} \in \tau \\ f^{-1}(Y) &= X \in \tau, \end{aligned}$$

so f is continuous.

f is not a homeomorphism because it is not open: indeed, $\{b\} \in \tau$ and

$$f(\{b\}) = \{y\} \notin \sigma.$$

Exercise 4. Consider the following relations. Are they equivalence relations? If yes, prove it. If not, find an example showing that one of the properties of equivalence relations fails.

1. On $X = \{\text{squares in } \mathbb{R}^2\}$, the relation *has the same area as*.

2. On $X = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ function}\}$, the relation $\sim$ given by $f \sim g$ if for every $x \in \mathbb{R}$, $f(x) \leq g(x)$. ⚠ **Correction in the definition of $\sim$.**

Solution. 1. The relation *has the same area as* on the set of squares in R^2 is an equivalence relation, because:

- for any square $A \in X$, A has the same area as A, i.e. the relation is reflexive;
- if $A, B \in X$ are so that A has the same area as B, then B has the same area as A, i.e. the relation is symmetric;
- if $A, B, C \in X$ are so that A has the same area as B and B has the same area as C, then A has the same area as C, i.e. the relation is transitive.

2. The relation $\sim$ is not an equivalence relation, because it is not symmetric: indeed, let $f \in X$ be the constant function $f(x) = 1$ for every $x \in \mathbb{R}$ and $g \in X$ the constant function $g(x) = 2$ for every $x \in \mathbb{R}$. Then $f \sim g$, because for every $x \in \mathbb{R}$ we have

$$f(x) = 1 \leq 2 = g(x),$$

but $g \not\sim f$ because

$$g(0) = 2 \not\leq 1 = f(0).$$