Proof of the theorem: fix ℓ; induction on d.

$d=2$: we know that $\exists$ 2-regular graph on $2m$ vertices and $c(G)\geq \ell$ $\forall m\geq \frac{\ell}{2}$

Note that here $\sum_{t=1}^{\ell-2}(d-1)^t = \sum_{t=1}^{\ell-2} 1 = \ell-2$ and $2(\ell-2)\geq \frac{\ell}{2}$.

$d-1 \rightsquigarrow d$: by induction, $\forall m \geq 2\sum_{t=1}^{\ell-2}(d-1)^t \left(\geq 2\sum_{t=1}^{\ell-2}(d-2)^t\right)$ $\exists G$ $(d-1)$-regular on $2m$ vertices and $c(G)\geq \ell$.

So $\mathcal{G} := \{\Gamma$ on $2m$ vertices s.t. $c(\Gamma)\geq \ell$ and $\forall v\in\Gamma$, $d-1\leq \deg(v)\leq d\} \neq \emptyset$

$\Rightarrow$ pick $G'\in\mathcal{G}$ w/ maximal number of edges.

Goal: G' is d-regular (then we are done)

Note: if G' has exactly one vertex of deg $d-1$, then

$$\underbrace{2|E(G')|}_{\text{even}} = \sum_{v\in G'}\deg_{G'}(v) = (2m-1)k+k-1 = \underbrace{2mk-1}_{\text{odd}} \quad \lightning$$

So it's enough to prove that G' has at most one vertex of degree $d-1$.

By contradiction, suppose $\exists v_1\neq v_2$ of degree $d-1$.

Claim 1: $\forall v\in G'$ s.t. $\deg_{G'}(v)=d-1$, $d_{G'}(v,v_i)\leq \ell-2$, $i=1,2$.

Indeed, if not, $\exists i$: $d_{G'}(v,v_i)\geq \ell-1$. Consider $G'\cup e$, where e is an edge between v_i and v.

Then $G'\cup e\in\mathcal{G}$, because

- by construction, every vertex has deg d or $d-1$
- if by contradiction $\exists$ nontrivial cycle γ of length $<\ell$, it needs to contain e (otherwise $\gamma\subseteq G'$, but $c(G')\geq \ell$) $\Rightarrow$ $\gamma\setminus e$ gives a path of length $<\ell-1$ between v and v_i $\lightning$

$\Rightarrow$ contradiction w/ the maximality of $|E(G')|$.

Claim 2: if $\deg(v)<d \Rightarrow |B(v,r)|\leq \sum_{t=0}^{r}(d-1)^t$.

↳ Left as an exercise
(similar to previous computations)

Then:

$$|B(v_1,\ell-2)\cup B(v_2,\ell-2)| = |B(v_1,\ell-2)|+|B(v_2,\ell-2)| - \underbrace{|B(v_1,\ell-2)\cap B(v_2,\ell-2)|}_{\geq 2 \text{ by Claim 1}} \overset{\text{Claim 2}}{\leq}$$

$$\leq 2\sum_{t=0}^{\ell-2}(d-1)^t - 2 = 2\sum_{t=1}^{\ell-2}(d-1)^t \leq m.$$

Let $\{v_1,\dots,v_p\} = B(v_1,\ell-2)\cup B(v_2,\ell-2)$ and $w_1,\dots,w_{2m-p}$ the other vertices. We have: $p\leq m$.
$\Rightarrow 2m-p\geq 2p-p=p$.

By claim 1, $\deg(w_j)=d$ $\forall j$.

Claim 3: there is at least one edge joining two w_j.

Indeed if not there are $\sum_{j=1}^{2m-p}\deg(w_j)=(2m-p)d$ edges between w_j's and v_i's.

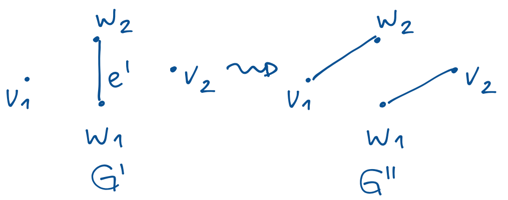
But the total number of edges emanating from the v_i is at most

$$\sum_{i=1}^{p} \deg(v_i) < pd \leq (2m-p)d$$

$\deg(v_1) = \deg(v_2) = d-1$

a contradiction.

Wlog, suppose e' is an edge between w_1 and w_2. Let G'' be the graph obtained by removing e' and adding two edges e_1 and e_2, where e_i joins v_i and w_i.



Then $G'' \in \mathcal{G}$:
- each vertex has degree d or $d-1$
- if $c(G'') < \ell$, $\exists$ non-trivial cycle $\Rightarrow$ it needs to contain e_1 and/or e_2
  $\Rightarrow$ one path of $\gamma \setminus \{e_1, e_2\}$ is a path between v_1 and w_1 or v_2 and w_2 or w_1 and w_2, and it has length $< \ell - 1$. But
  - $d_{G'}(v_i, w_i) \geq \ell - 1$ (since $w_i \notin B(v_i, \ell - 2)$)
  - $d_{G' \setminus e'}(w_1, w_2) \geq \ell - 1$, otherwise there is a cycle in G' of length $< \ell$

    $\quad \| $

    $d_{G'' \setminus (e_1 \cup e_2)}(w_1, w_2)$

  so we get a contradiction, i.e. $c(G'') \geq \ell$.

But $|E(G'')| > |E(G)|$, a contradiction.

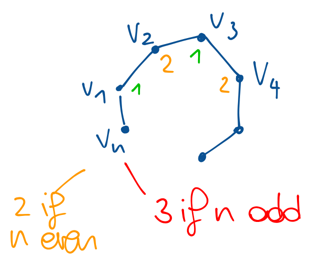
<u>Note</u>: the proof doesn't give an explicit construction. □

In general, it is often hard to explicitly construct examples of graphs (or other objects) with extremal properties. Still, one can sometimes show they exist using other methods, one of which is the <u>probabilistic method</u>, which is the following: define a probability measure on a set of graphs and show that the probability that a graph has the desidered property is positive.

As an example, let's talk about chromatic numbers.

An <u>n-coloring</u> of a graph G is a map $c: V(G) \longrightarrow \{1, \dots, n\}$ s.t. if $b(e) = \{v_1, v_2\}$, then $c(v_1) \neq c(v_2)$. G is <u>n-colorable</u> if it admits an n-coloring.

<u>Ex</u>. G bipartite $\Longleftrightarrow$ 2-colorable
- C_n is $\begin{cases} \text{2-colorable}, & n \text{ even} \\ \text{3-colorable}, & n \text{ odd} \end{cases}$



The chromatic number of a graph G is $\chi(G) := \min\{n \mid G \text{ is } n\text{-colorable}\}$.

Ex. $\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}$

$\chi(K_n) = n$

$\chi(\text{tree}) = \begin{cases} 1 & \text{if } |V|=1 \\ 2 & \text{otherwise} \end{cases}$

Rmk: G has large girth $\Rightarrow$ G looks like a tree, locally

? Does that imply that graphs of large girth have small χ?

No!

Thm (Erdős)

$\forall k \in \mathbb{N}$ $\exists$ graph G with $c(G) > k$ and $\chi(G) > k$.

Idea of the proof: $V = \{1, \dots, n\}$, $p \in]0,1[$, $\mathcal{G}(n,p) := \{G \text{ simple graph} : V(G) = V\}$

To give a probability: $\mathbb{P}_{n,p}(G \text{ contains a given edge}) = p$

$\mathbb{P}_{n,p}(G) = p^m (1-p)^{\binom{n}{2}-m}$, where $m = |V(G)|$

Show that, for $p = n^{\varepsilon - 1}$, $\varepsilon \in]0, 1/k[$:

$$\mathbb{P}_{n,p}\left(\begin{array}{l} G \text{ contains} < \frac{n}{2} \text{ cycles of length} \leq k \\ \text{and} < \frac{n}{2k} \text{ indep vertices} \end{array}\right) > 0 \quad \text{as } n \gg 0$$

no two are adjacent

$\Rightarrow$ $\exists H \in \mathcal{G}(n,p)$ with $< \frac{n}{2}$ cycles of length $\leq k$ and so that every set of indep. vertices has size $< \frac{n}{2k}$.

Let G be obtained from H by removing one vertex per cycle, together with all the edges incident to these vertices.

$\Rightarrow$ G contains no cycle of length $\leq k$, so $c(G) > k$.

Rmk: if c is an m-coloring, $V(G) = c^{-1}(1) \sqcup \dots \sqcup c^{-1}(m)$ and no two vertices in $c^{-1}(i)$ can be adjacent, i.e. $\forall i$, $c^{-1}(i)$ is a set of independent vertices $\Rightarrow |c^{-1}(i)| < \frac{n}{2k}$ (because vertices of G are adj in G iff they are adj in H, so $c^{-1}(i)$ is a set of indep. vertices in H as well).

$$\Rightarrow m \geq \frac{|V(G)|}{\max_i |c^{-1}(i)|} > \frac{|V(H)| - \frac{n}{2}}{n/2k} = \frac{n/2}{n/2k} = k$$

$\Rightarrow \chi(G) > k$.

$\square$

Other application: Ramsey numbers

Fact: at a party of 6 people there are either 3 people who mutually know each other, or 3 people who don't know each other.

$\rightarrow$ graph: vertices = people, edge: two people who know each other

G simple graph on 6 vertices $\Rightarrow$ either $\exists$ 3 vertices all adjacent to each other, or $\exists$ 3 vertices which have no edges between each other.

$G \supseteq K_3$ G contains 3 independent vertices

Thm (Ramsey)
$\forall r \geq 2$ $\exists n \in \mathbb{N}$ s.t. $\forall$ simple graph G on at least n vertices, either $G \supseteq K_r$ or G contains r indep. vertices.

$R(k) := \min\{n \mid \forall G, |V(G)| \geq n, \text{ either } G \supseteq K_k \text{ or } G \supseteq k \text{ indep. vertices}\}$ Ramsey number

The theorem tells us that $R(k) < \infty$ (and one can actually give an upper bound). Moreover: $R(3) = 6$, $R(4) = 18$, $R(k)$, $k \geq 5$: unknown. Lower bound ?

Thm (Erdős)
$\forall k \geq 3$, $R(k) > 2^{k/2}$.

Goal: Show that $\forall n \leq 2^{k/2}$ $\exists$ G simple graph on n vertices which doesn't contain K_k nor any set of k indep. vectors.

Show: $\mathbb{P}(G \text{ has } k \text{ indep. vertices}) < \frac{1}{2}$, $\mathbb{P}(G \supseteq K_k) < \frac{1}{2}$

$\Rightarrow \mathbb{P}(G \text{ has no } k \text{ indep. vertices and } G \not\supseteq K_k) > 0$.