More on random graphs

Once we fix a model for random graphs, quantities such as the number of edges or the diameter became <u>random variables</u>. In particular, we can compute their <u>expected value</u>.

$X : \mathcal{G}(n) \longrightarrow \mathbb{R} \implies \mathbb{E}_{n,p}(X) := \sum_{G \in \mathcal{G}(n)} X(G) \cdot \mathbb{P}_{n,p}(G).$

Remember that the expected value is linear: X, Y random variables, $\alpha, \beta \in \mathbb{R}$

$$\implies \mathbb{E}_{n,p}(\alpha X + \beta Y) = \alpha \mathbb{E}_{n,p}(X) + \beta \mathbb{E}_{n,p}(Y).$$

Let's look at an example:

<u>Lemma</u>

$X : \mathcal{G}(n) \longrightarrow \mathbb{R},\ X(G) = |E(G)| \implies \mathbb{E}_{n,p}(X) = \binom{n}{2} p.$

We will give two proofs:

<u>Proof 1</u>: using the definition of $\mathbb{E}_{n,p}$ and setting $N := \binom{n}{2}$:

$$\mathbb{E}_{n,p}(X) = \sum_{G \in \mathcal{G}(n)} X(G) \cdot \mathbb{P}_{n,p}(G) = \sum_{G \in \mathcal{G}(n)} |E(G)| \cdot \mathbb{P}_{n,p}(G) = \sum_{i=0}^{N} \sum_{\substack{G \in \mathcal{G}(n): \\ |E(G)| = i}} i \cdot \mathbb{P}_{n,p}(G) =$$

$$= \sum_{i=0}^{N} \sum_{\substack{G \in \mathcal{G}(n): \\ |E(G)| = i}} i\, p^i (1-p)^{N-i} \underset{\substack{\uparrow \\ |\{G : |E(G)| = i\}| = \binom{N}{i}}}{=} \sum_{i=0}^{N} \binom{N}{i} i\, p^i (1-p)^{N-i} = \sum_{i=1}^{N} \binom{N}{i} i\, p^i (1-p)^{N-i}$$

We have: $\binom{N}{i} i = \frac{N!}{i!(N-i)!} \cdot i = N \frac{(N-1)!}{(i-1)!(N-i)!} = N \binom{N-1}{i-1}$

So $\mathbb{E}_{n,p}(X) = \sum_{i=1}^{N} N \binom{N-1}{i-1} p\, p^{i-1} (1-p)^{(N-1)-(i-1)} = Np \sum_{J=1}^{N-1} \binom{N-1}{J} p^J (1-p)^{N-1-J} = Np\,(p + (1-p))^{N-1} = Np.$

$\square$

<u>Proof 2</u>: we use the linearity of $\mathbb{E}_{n,p}$. Define the random variables X_e, $e \in [V]^2$, given by $X_e(G) = \begin{cases} 1 & \text{if } e \in E(G) \\ 0 & \text{otherwise} \end{cases}$.

$\implies X = \sum_{e \in [V]^2} X_e$ and thus $\mathbb{E}_{n,p}(X) = \sum_e \mathbb{E}_{n,p}(X_e)$.

Moreover, $\forall e \in [V]^2$:

$$\mathbb{E}_{n,p}(X_e) = \sum_{G \in \mathcal{G}(n)} X_e(G) \cdot \mathbb{P}_{n,p}(G) \underset{\substack{\uparrow \\ X_e(G) \neq 0 \\ \text{only if } e \in E(G)}}{=} \sum_{G : e \in E(G)} 1 \cdot \mathbb{P}_{n,p}(G) = \mathbb{P}_{n,p}(G \text{ contains } e) = p$$

$\implies \mathbb{E}_{n,p}(X) = \sum_e p = \binom{n}{2} p.$

$\square$

Computing expected values can be useful to estimate probabilities:

<u>Lemma</u> (Markov's inequality)

$X : \mathcal{G}(n) \longrightarrow \mathbb{R}_{\geq 0}$ random variable, $a > 0$. Then

$$\mathbb{P}_{n,p}(X \geq a) \leq \mathbb{E}_{n,p}(X)/a.$$

<u>Proof</u>: $\mathbb{E}_{n,p}(X) = \sum_G X(G) \cdot \mathbb{P}_{n,p}(G) \underset{\substack{\uparrow \\ X \geq 0}}{\geq} \sum_{G : X(G) \geq a} X(G) \cdot \mathbb{P}_{n,p}(G) \geq a \sum_{G : X(G) \geq a} \mathbb{P}_{n,p}(G) = a\, \mathbb{P}_{n,p}(X \geq a).$ $\square$

This can be used to show results like the following.

Prop.
Fix $k \geq 3$, $\varepsilon \in]0, 1/k[$, $p = p(n) := n^{\varepsilon - 1}$.

$$\mathbb{P}_{n,p}\left(G \text{ has} \geq \tfrac{n}{2} \text{ nontrivial cycles of length} \leq k\right) \xrightarrow[n\to\infty]{} 0.$$

Proof. We first compute the expected number of cycles of length $\leq k$:
Fix C a cycle $(v_0, e_1, v_1, \dots, e_i, v_i = v_0)$, $i \leq k$. Then

$$\mathbb{P}_{n,p}(G \text{ contains } C) = \mathbb{P}_{n,p}(E(G) \supseteq \{e_1, \dots, e_i\}) = p^i.$$

$X_C : \mathcal{G}(n) \longrightarrow \mathbb{R}$ given by $X_C(G) = \begin{cases} 1 & \text{if } G \text{ contains } C \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow \mathbb{E}_{n,p}(X_C) = \mathbb{P}_{n,p}(G \text{ contains } C) = p^i$
Moreover, if $X : \mathcal{G}(n) \longrightarrow \mathbb{R}_{\geq 0}$ is given by $X(G) = \#$ cycles of length $\leq k$, $X = \sum_{\substack{C \text{ cycle} \\ \text{of length} \leq k}} X_C$

$$\Rightarrow \mathbb{E}_{n,p}(X) = \sum_{i=3}^{k} \sum_{\substack{C \text{ cycle} \\ \text{of length } i}} \mathbb{E}_{n,p}(C)$$

nontrivial cycles
in simple graphs
have length ≥ 3

Rmk: cycle of length i $\longleftrightarrow$ ordered list of i vertices up to cyclic permutation & reversing the order

$\Rightarrow \#$ cycles of length i = $\#$ ordered lists of i vertices up to cyclic permutation & reversing the order =

$$= \frac{n(n-1)\cdots(n-i+1)}{2i}$$

$$\Rightarrow \mathbb{E}_{n,p}(X) = \sum_{i=3}^{k} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i$$

So, by Markov's inequality:

$$\mathbb{P}_{n,p}\left(X \geq \tfrac{n}{2}\right) \leq \frac{\mathbb{E}_{n,p}(X)}{n/2} = \frac{2}{n} \sum_{i=3} \frac{\overbrace{\boxed{n(n-1)\cdots(n-i+1)}}^{\leq n^i}}{\underbrace{\boxed{2i}}_{\geq 2}} p^i \leq \frac{1}{2n} \sum_{i=3}^{k} \underbrace{\boxed{n^i (n^{\varepsilon - 1})^i}}_{= (n^\varepsilon)^i \leq n^{\varepsilon k}} \leq$$

$$\leq \frac{(k-2)}{2} n^{\varepsilon k - 1} \xrightarrow[n\to\infty]{} 0 \text{ since by assumption } \varepsilon k - 1 < 0.$$

$\square$

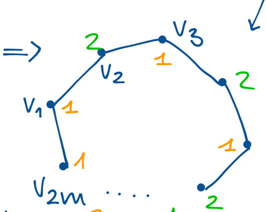
This is the first step towards another theorem proven by Erdős using the probabilistic method:

Thm
$\forall k \; \exists$ graph G with girth $> k$ and chromatic number $> k$.

Here: the girth of a graph is $g(G) := \min\{n \mid G \text{ contains a nontr. } n\text{-cycle}\}$ and the chromatic number is $\chi(G) := \min\{n \mid \exists\, n\text{-coloring of } G\}$, where an n-coloring of G is a map $c : V(G) \longrightarrow \{1, \dots, n\}$ s.t. $c(v_1) \neq c(v_2)$ if v_1 and v_2 are adjacent. G is n-colorable if it admits an n-coloring (i.e. $\chi(G) \leq n$).

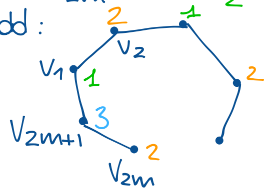
Ex: $g(C_n)=n$, but $\chi(C_n)=\begin{cases}2 & n \text{ even}\\ 3 & n \text{ odd}\end{cases}$

n even $\Rightarrow$



$\chi(C_n)\leq 2$ and $\chi(G)\neq 1$ (there are adj. vertices)

n odd:



$\Rightarrow \chi(C_n)\leq 3$; if there were a 2-coloring c, wlog $c(v_1)=1 \Rightarrow c(v_i)=\begin{cases}1 & \text{if } i \text{ odd}\\ 2 & \text{if } i \text{ even}\end{cases} \Rightarrow c(v_{2m+1})=1=c(v_1)$ ↯

$\Rightarrow \chi(C_n)=3.$

Ex. $K_n, n\geq 3$: any K_n contains a 3-cycle $\Rightarrow g(K_n)=3$

any two vertices are adjacent $\Rightarrow$ they all need to have different colors $\Rightarrow \chi(K_n)=n$.

Rmk: $\chi(G)=1 \Leftrightarrow$ all vertices are isolated (have deg 0)

G 2-colorable $\Leftrightarrow$ bipartite

In particular, since trees are bipartite (they contain no odd cycles), they are 2-colorable.

Rmk: large girth $\Rightarrow$ locally, G looks like a tree, which is 2-colorable. Erdős's result tells us that globally the situation might be very different.

Proof of Erdős's thm

Fix $k\geq 3$, $\varepsilon\in\,]0,1/k[$, $p=n^{\varepsilon-1}$.

We have seen that $P_{n,p}(G \text{ contains } r \text{ indep. vertices})\leq \binom{n}{r}(1-p)^{\binom{r}{2}}$

$$\Rightarrow P_{n,p}(G \text{ contains } r \text{ indep. vertices})\leq n^r(1-p)^{r(r-1)/2}=\left(n(1-p)^{r-1/2}\right)^r\leq$$

$$\leq\left(ne^{-p(r-1)/2}\right)^r$$

$1-p\leq e^{-p}\ \forall p$

Now one can show that for our choice of p and for $r\geq \frac{n}{2k}$, the bound $\longrightarrow 0$ as $n\to\infty$:

$$P_{n,p}\left(G \text{ cont. } \tfrac{n}{2k} \text{ indep. vertices}\right)\longrightarrow 0 \text{ as } n\to\infty$$

Since $P_{n,p}(G \text{ contains} \geq \frac{n}{2} \text{ cycles of length} \leq k)\longrightarrow 0$ as $n\to\infty$, $\exists n$ s.t.

$$P_{n,p}(G \text{ cont. } \tfrac{n}{2k} \text{ indep. vert}),\ P_{n,p}(G \ni \geq \tfrac{n}{2} \text{ cycles of length} \leq k)<\tfrac{1}{2}.$$

$\exists G\in\mathcal{G}(n)$ s.t. (1) G has $<\frac{n}{2}$ cycles of length $\leq k$

(2) if U is a set of indep. vertices, $|U|<\frac{n}{2k}$.

Modify H by removing one vertex per cycle (and the edges adjacent to those vert.)

Rmk: $\forall$ graph K, if c is a k-coloring, $V(K)=c^{-1}(1)\sqcup\ldots\sqcup c^{-1}(k)$. Moreover, any two vert. in the same $c^{-1}(i)$ cannot be adjacent, i.e. the vertices of $c^{-1}(i)$ are indep. $\forall i$

If $\alpha(K):=\max \#$ of indep. vertices $\Rightarrow |V(K)|\leq k\cdot\alpha(K)$, so $k\geq\frac{|V(K)|}{\alpha(K)} \Rightarrow \chi(K)\geq\frac{|V(K)|}{\alpha(K)}$.

Then: • H contains no cycle of length $\leq k \Rightarrow g(H)>k$

• $\chi(H)\geq\frac{\boxed{|V(H)|}\geq |V(G)|-\frac{n}{2}=\frac{n}{2}}{\boxed{\alpha(H)}\leq\alpha(G)<n/2k} \Rightarrow \chi(H)\geq\frac{n}{2}\cdot\frac{2k}{n}=k.$

□