Note that the converse of the previous proposition is not true:

Ex. $G =$  $\Rightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which has 3 eigenvalues $(1, 0, -1)$ of multiplicity 1

So $\lambda_1 < \lambda_2$ but G is not connected.

On the other hand, if we assume that G is finite and d-regular, then G is conn. iff $\lambda_1 < \lambda_2$. We will see this by looking at another matrix associated to G (the Laplacian).

Prop.
G finite, without loops. The following are equivalent:
(a) G bipartite;
(b) $\mathrm{spec}(A) = \mathrm{spec}(-A)$ (i.e. if $\alpha \in \mathrm{spec}(A)$, $-\alpha \in \mathrm{spec}(A)$ and α and $-\alpha$ have the same multiplicity).
(c) $\forall \ell$ odd, $\mathrm{tr}(A^\ell) = 0$.

Proof.
(a) $\Rightarrow$ (b) Let $\{L = \{v_1, \dots, v_k\}, R = \{v_{k+1}, \dots, v_n\}\}$ be a bipartition. Then

$$A = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

where $B_{ij} = \#$edges between v_i and v_{k+j} $\Rightarrow B$ is a $k \times (n-k)$ matrix.

Supp. $\alpha \in \mathrm{spec}(A)$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ eigenvector for α, where f_1 has n-k entries and f_2 has k entries

Ex 12 Show that $Af' = -\alpha f'$, where $f' = \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}$.

$\Rightarrow -\alpha$ is an eigenvector.

Moreover, $f^{(1)} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \end{pmatrix}, \dots, f^{(m)} = \begin{pmatrix} f_1^{(m)} \\ f_2^{(m)} \end{pmatrix}$ are indep. eigenvectors for α $\Leftrightarrow$

$\begin{pmatrix} f_1^{(1)} \\ -f_2^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} f_1^{(m)} \\ -f_2^{(m)} \end{pmatrix}$ are independent eigenvectors for $-\alpha$

$\Rightarrow \alpha$ and $-\alpha$ have the same multiplicity.

(b) $\Rightarrow$ (c) $$\mathrm{tr}(A^\ell) = \sum_{\beta \in \mathrm{spec}(A^\ell)} \beta = \sum_{\alpha \in \mathrm{spec}(A)} \alpha^\ell \underset{\mathrm{spec}(A) = -\mathrm{spec}(A)}{=} \frac{1}{2}\Big(\sum_{\alpha \in \mathrm{spec}(A)} \alpha^\ell + \sum_{\alpha \in -\mathrm{spec}(A)} \alpha^\ell\Big) = \frac{1}{2} \sum_{\alpha \in \mathrm{spec}(A)} (\alpha^\ell + (-\alpha)^\ell) \underset{\ell \text{ odd}}{=} 0$$

(c) $\Rightarrow$ (a) By ex. 11, there are no closed walks of odd length, and we've seen that this implies that G is bipartite.

$\square$

The <u>Laplacian</u> of a graph G is the matrix

$$L = D - A$$

where $D = \mathrm{diag}(\deg(v_1), \dots, \deg(v_n))$.

Again, L is real and symmetric, so it has real eigenvalues $\mu_n \geq \dots \geq \mu_1$.

<u>Ex. 13</u> Prove that v_i is an isolated vertex (i.e. there are no edges between v_i and $v_j \neq v_i$) $\iff L_{ii} = 0$.

<u>Rmk</u>: G d-regular $\Rightarrow L = d\,\mathrm{Id} - A \Rightarrow$ easy to go from $\mathrm{spec}(A)$ to $\mathrm{spec}(L)$.

To $e \in E$, e not a loop, we associate the $n \times n$ matrix L_e given by:

$$(L_e)_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in b(e) \\ -1 & \text{if } b(e) = \{v_i, v_j\} \\ 0 & \text{otherwise} \end{cases}$$

<u>Lemma</u>

$$L = \sum_{\substack{e \text{ not} \\ \text{loop}}} L_e .$$

<u>Proof</u>:

$$\Big(\sum_{e \in E} L_e\Big)_{ii} = \sum_{\substack{e \text{ not} \\ \text{loop}, \\ v_i \in b(e)}} 1 = \#\{e \in E \mid b(e) \ni v_i, e \text{ not loop}\} = \deg(v_i) - \#\{e \in E \mid b(e_i) = \{v_i\}\} =$$

$$= (D)_{ii} - A_{ii}$$

If $i \neq j$:

$$\Big(\sum_{\substack{e \text{ not} \\ \text{loop}}} L_e\Big)_{ij} = \sum_{\substack{e \in E, \\ b(e) = \{v_i, v_j\}}} (-1) = -\#\{e \in E \mid b(e) = \{v_i, v_j\}\} = -A_{ij} = (D)_{ij} - A_{ij} \quad \text{(since } D \text{ is diagonal).}$$

$\square$

<u>Rmk</u>: this shows that the Laplacian "doesn't see" loops — if two graphs are the same except for the loops, they have the same laplacian.

<u>Thm</u>

(1) L is positive semidefinite, i.e. $\forall f \in \mathbb{R}^n$, $\langle f, Lf \rangle \geq 0$. In particular, $\mu_1 \geq 0$.

(2) $\mu_1 = 0$.

(3) The multiplicity of the 0 eigenvalue is the number of connected comp. of G.
In particular, G connected $\iff \mu_2 > 0$.

Note that if G is d-regular, G is connected if and only if $\mu_2 > 0$ if and only if $\lambda_1 > \lambda_2$.

<u>Proof</u>.

(1) Let e be an edge, not a loop, with $b(e) = \{v_{i_0}, v_{i_1}\}$. Then:

$(L_e)_{i_0 i_0} = (L_e)_{i_1 i_1} = 1$, $(L_e)_{i_0 i_1} = (L_e)_{i_1 i_0} = -1$ and the other entries are zero, so

$$\langle f, L_e f \rangle = \sum_{i=1}^{n} f_i \Big(\sum_{j=1}^{n} (L_e)_{ij} f_j \Big) = f_{i_0}^2 + f_{i_1}^2 - 2 f_{i_0} f_{i_1} = (f_{i_0} - f_{i_1})^2 \geq 0.$$

So each L_e is positive semidefinite and thus so is L.

If $\mu \in \mathrm{spec}(L)$ and f is a nonzero eigenvector $\Rightarrow 0 \leq \langle f, Lf\rangle = \langle f, \lambda f\rangle = \lambda \underbrace{\langle f, f\rangle}_{\substack{\vee \\ 0}}$

$$\Rightarrow \lambda \geq \frac{\langle f, Lf\rangle}{\langle f, f\rangle} \geq 0.$$

(2) f vector with all entries $1 \Rightarrow (Lf)_i = \sum_{j=1}^{n} L_{ij} f_j = \sum_{j=1}^{n} L_{ij} = \deg(v_i) - \sum_{j=1}^{n} A_{ij} = 0$

$$\Rightarrow Lf = 0 \cdot f, \text{ i.e. } \mu_1 = 0.$$

(3) Supp. G has k connected components $C_1, \dots, C_k$.

We will prove that there are k linearly independent eigenvectors for 0 which span the eigenspace of 0. This implies that the multiplicity of 0 is k.

$\forall i$, let $f^{(i)}$ be the vector $(f^{(i)})_j = \begin{cases} 1 & \text{if } v_j \in V(C_i) \\ 0 & \text{otherwise} \end{cases}$

<u>Rmk</u>: since $V \quad , \sqcup \dots \sqcup V(C_k) = V$, $\forall j \ \exists!\, i_j$ s.t. $(f^{(i_j)})_j \neq 0$

$$\Rightarrow \text{if } \sum_{i=1}^{n} \alpha_i f^{(i)} = 0 \Rightarrow \Big(\sum_{i=1}^{n} \alpha_i f^{(i)}\Big)_j = 0 \ \forall j \Rightarrow \alpha_j \underbrace{(f^{(i_j)})_j}_{1} = 0 \ \forall j \Rightarrow \alpha_j = 0 \ \forall j, \text{ i.e.}$$

$f^{(1)}, \dots, f^{(k)}$ are lin. indep.

Moreover:

$$(Lf^{(i)})_j = \sum_{\ell=1}^{n} (L_{j\ell})(f^{(i)})_\ell = \sum_{\substack{\ell: \\ v_\ell \in C_i}} L_{j\ell}$$

- if $v_j \notin C_i \Rightarrow j \neq \ell \ \forall \ell : v_\ell \in C_i$ and $L_{j\ell} = -A_{j\ell} = 0$ since there are no edges between vertices in different components

- if $v_j \in C_i$: $(Lf^{(i)})_j = L_{jj} + \sum_{\substack{\ell \neq j, \\ v_\ell \in C_i}} L_{j\ell} = \deg(v_j) - \sum_{\substack{\ell: \\ v_\ell \in C_i}} A_{j\ell} = 0$

$\Rightarrow (Lf^{(i)})_j = 0 \ \forall j$, i.e. $Lf^{(i)} = 0$, i.e. $f^{(i)}$ is an eigenvector for 0.

If $f \neq 0$ is such that $Lf = 0$, then

$$0 = \langle f, Lf\rangle = \sum_{e \in E} \langle f, L_e f\rangle = \sum_{i<j} \sum_{\substack{e: \\ b(e) = \{v_j, v_i\}}} (f_j - f_i)^2$$

$\Rightarrow f_j = f_i$ if v_j and v_i are adjacent; by induction, this implies that if v_j and v_i are joined by a walk, $f_j = f_i$.

As each C_i is connected, any two vertices in C_i are joined by some walk, so f_j is the same $\forall j : v_j \in C_i$. Say $f_j = \alpha_i \ \forall j : v_j \in C_i$

$\Rightarrow f = \sum_{i=1}^{k} \alpha_i f^{(i)}$, i.e. the $f^{(i)}$ generate the eigenspace of 0.

Now, G conn. $\Leftrightarrow$ G has $1!$ conn. component $\Leftrightarrow$ the multiplicity of 0 is 1 $\Leftrightarrow \mu_2 > 0$. $\square$

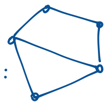
Girth of a graph

G (finite) graph; its girth is

$$c(G) := \inf\{\ell(\gamma) \mid \gamma \text{ nontrivial cycle in } G\}.$$

By definition, $c(G)=\infty$ if $\nexists$ nontrivial cycles, i.e. if G is a forest.

Ex.

- G_1:  $c(G_1)=3$

- G_2 a cycle on n vertices: $c(G_2)=n$

- $c(K_n)=3$

Ex. 13 $c(G)=1 \iff G$ contains a loop and $c(G)\geq 3 \iff G$ is simple.

The cycle example shows that we can have arbitrarily large girth if we have "few" edges with respect to the number of vertices (G not tree $\Rightarrow |E(G)|\geq|V(G)|$, if G is connected. But if we add edges to a cycle, the girth decreases).

So the question is: for a conn. graph with n vertices and m edges, how large can $c(G)$ be?

Recall: G d-regular w/ n vertices $\Rightarrow$ the number of edges is determined ($nd/2$).

So for d-regular graphs, the question is: if G is conn. d-regular w/ n vertices, how large can $c(G)$ be?

Ex. $d=0 \Rightarrow |E(G)|=0 \underset{G \text{ conn.}}{\Rightarrow} |V(G)|=1 \Rightarrow G$ tree $\Rightarrow c(G)=\infty$

$d=1 \underset{\text{Hand-shaking lemma}}{\Rightarrow} |V(G)|=2|E(G)|$. If $|V(G)|>2$, it cannot be connected. So $|V(G)|=2$, $G=$

$\Rightarrow$ tree $\Rightarrow c(G)=\infty$.

$d=2 \Rightarrow$ by ex. 8, it is a cycle, so $c(G)=|E(G)|$.

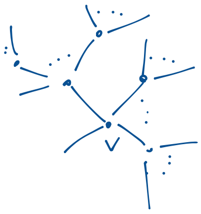
So the cases $d\leq 2$ are not interesting.

What about $d\geq 3$?

Prop.
G finite, d-regular graph, $d\geq 3$, on n vertices. Then

$$c(G)\leq 2\log_{d-1}\left(\frac{d-2}{d}(n-1)+1\right).$$

Proof. If G contains a loop, $c(G)=1$ and we are done. So suppose G doesn't contain any loop. For $v\in V(G)$, we want to count the number of non-backtracking walks starting at v and length $\leq k$. We have:



First edge: d choices
Second edge: d−1 choices (non-backtracking)
Same for the n^{th} edge, $\forall n \geq 2$

$$\Rightarrow d(d-1)^{n-1} = \#\text{ walks of length } = n \geq 1$$

$$\Rightarrow \#\text{ walks of length} \leq k = 1 + \sum_{n=1}^{k} d(d-1)^{n-1} \underset{\substack{\uparrow \\ \text{geometric series}}}{=} 1 + d\,\frac{(d-1)^k - 1}{d-2}$$

If $2k < c(G)$, then all the endpoints of these walks are distinct (otherwise we can find a nontrivial cycle of length $< c(G)$), so if $k < \frac{c(G)}{2}$

$$1 + \frac{d}{d-2}\left[(d-1)^k - 1\right] \leq n$$

i.e. $(d-1)^k \leq (n-1)\left(\frac{d-2}{d}\right) + 1$, i.e. $k \leq \log_{d-1}\left(\left(\frac{d-2}{d}\right)(n-1)+1\right) - 1$.

If $c(G)$ even $\rightarrow$ we can choose $k = \frac{c(G)}{2} - 1$
If $c(G)$ odd $\rightarrow$ we can choose $k = \frac{c(G)-1}{2}$
$\Rightarrow$ we can always choose $k \geq \frac{c(G)}{2} - 1$

$$\Rightarrow \frac{c(G)}{2} - 1 \leq \log_{d-1}\left(\frac{d-2}{d}(n-1)+1\right) - 1$$

$$\text{i.e. } c(G) \leq 2\log_{d-1}\left(\frac{d-2}{d}(n-1)+1\right).$$

$\square$

How close can we get to this bound?

<u>Thm</u> (Erdős–Sachs)
$\forall d \geq 2, \forall \ell \geq 4, \forall m \geq 2\sum_{t=1}^{\ell-2}(d-1)^t$, $\exists$ d-regular graph G on $2m$ vertices with $c(G) \geq \ell$.

So if $m = 2\sum_{t=1}^{\ell-2}(d-1)^t = 2\left[\frac{(d-1)^{\ell-1}-1}{d-2} - 1\right]$, then $(d-1)^{\ell-1} = \left(\frac{m}{2}+1\right)(d+2)+1$, i.e.

$$c(G) \geq \ell = \log_{d-1}\left[(d+2)\left(\frac{m}{2}+1\right)+1\right] + 1$$

So for d fixed, $n \to \infty$, the upper bound is $\sim 2\log_{d-1}(n)$ and the lower bound is $\sim \log_{d-1}(n)$.

$\rightsquigarrow$ same order of growth (we can't do better than logarithmic). Still, it's unknown what's the right constant in front of $\log_{d-1}(n)$.