Ramsey numbers

For this section we will consider only simple graphs, i.e. graphs without loops or multiple edges. In this case, we can identify edges with the pair of endpts, so we can describe a graph as a pair (V,E), where $E \subseteq [V]^2 = \{\text{subsets of } V \text{ of card. } 2\}$.

Given $(V,E,b) \rightsquigarrow (V, \{b(e) \mid e \in E\})$; conversely, $(V,E) \rightsquigarrow (V,E,b)$ where $b(e)=e$.

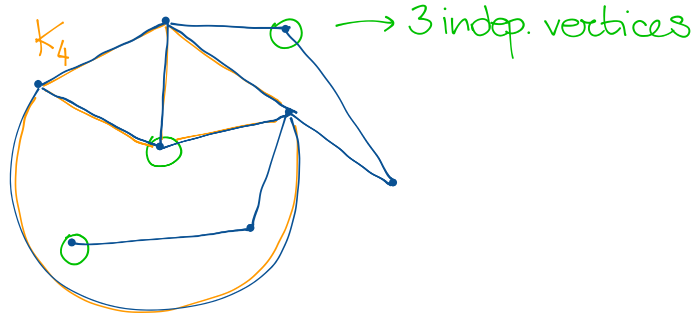
We will be interested in the following result:

Thm (Ramsey)

$\forall r \geq 2\ \exists n \in \mathbb{N}$ s.t. $\forall G$ simple graph with $|V(G)| \geq n$ contains either a complete graph K_r or r independent vertices.

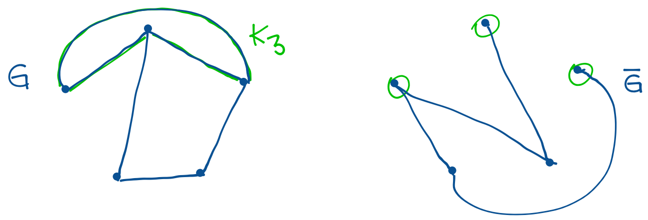
Here, r vertices $\{v_1, \dots, v_r\}$ are independent if no two are adjacent.

Ex.



Note: $G=(V,E)$ simple graph $\Rightarrow$ we can define $\bar{G} = (V, [V]^2 \setminus E)$ the complement of G, i.e. the graph with the same vertices and the complementary set of edges.

Ex.



An independent set in G corresponds to a complete graph in $\bar{G}$.

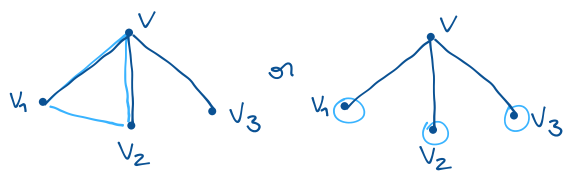
We denote by $R(r) := \min\{n \mid \forall G \text{ simple}, |V(G)| \geq n, \text{ either } G \supseteq K_r \text{ or } G \supseteq r \text{ indep. vertices}\}$.

Ramsey's thm is then equivalent to saying that $\forall r \geq 2, R(r) < \infty$.

Ex. At a party of 6, there either are 3 people who all know each other, or 3 people who don't know each other : $R(3)=6$.

$\hookrightarrow$ $V=$ people at the party, edge if they know each other

Proof: let G be a simple graph on ≥ 6 vertices and pick $v \in V$. Supp. $\deg(v) \geq 3$ and consider three edges $\{v,v_1\}, \{v,v_2\}, \{v,v_3\}$. If G contains at least one among $\{v_1,v_2\}, \{v_2,v_3\}$ and $\{v_1,v_3\}$, then G contains a K_3. If it doesn't contain any, $\{v_1,v_2,v_3\}$ are 3 indep. vertices.



If $\deg(v) < 3$, look at $\bar{G} \Rightarrow \deg_{\bar{G}}(v) \geq 3 \Rightarrow$ by the same argument either $\bar{G} \supseteq K_3$ or $\bar{G} \supseteq 3$ indep. vertices.

$\bar{G} \supseteq K_3 \Leftrightarrow G \supseteq 3$ indep. vertices

$\bar{G} \supseteq 3$ indep. vertices $\Leftrightarrow G \supseteq K_3$

$\Rightarrow R(3) \leq 6$.

Ex. 12 Show that $K(3) \geq 6$ (construct a graph on 5 vertices which doesn't contain a K_3 nor 3 indep. vertices).

To prove Ramsey's theorem, we generalize the notion slightly: $\forall r, s \geq 2$, set

$$R(r,s) := \min\{n \mid \forall G \text{ simple}, |V(G)| \geq n, \text{ either } G \supseteq K_r \text{ or } G \supseteq s \text{ indep vertices}\}.$$

Clearly, $R(r) = R(r,r)$.

We will prove:

Thm

$R(r,s) < \infty \ \forall r,s \geq 2$.

Proof:

Claim 1: $R(2,n) = R(n,2) = n$.

Claim 2: $\forall r,s \geq 3$, if $R(r-1,s)$ and $R(r,s-1)$ are finite, $R(r,s) \leq R(r-1,S) + R(r,s-1)$.

Then by repeatedly applying claim 2, we get a bound on $R(r,s)$ in terms of a finite sum of $R(2,m)$ or $R(m,2)$ and by claim 1 we are done.

Proof of claim 1:

If $G = (V,E)$ is the graph with $E = \emptyset \Rightarrow G$ can only contain a set of $|V|$ indep. vertices $\Rightarrow R(2,n) \geq n$.

If G has $|V| \geq n$, either $\exists$ edge, i.e. a K_2, or all vertices are independent $\Rightarrow n \geq R(2,n)$.

$\square$

Proof of claim 2:

Let G be a graph with $|V| \geq R(r-1,S) + R(r,s-1)$; our goal is to show that it contains either a K_r or s indep. vertices.

Pick $v \in V$ and let $V_1 := \{w \in V \mid w \text{ is adjacent to } v\}$, $V_2 := V \setminus V_1$.

Since $|V_1| + |V_2| = |V|$, either $|V_1| \geq R(r-1,S)$ or $|V_2| \geq R(r,s-1)$. Assume $|V_1| \geq R(r-1,S)$, the other case being analogous.

Look at $G_1 = (V_1, E \cap [V_1]^2)$, i.e. the graph containing the vertices in V_1 and all edges with endpoints in V_1. $|V_1| \geq R(r-1,S) \Rightarrow$ either $G_1 \supseteq K_S$, but then $G \supseteq K_S$, or G_1 contains $r-1$ indep. vertices $\{v_1, \dots, v_{r-1}\} \Rightarrow \{v, v_1, \dots, v_{r-1}\}$ are r indep. vertices in G.

$\square$

From this theorem we can also compute upper bounds on $R(r)$, but it is a hard problem to compute these numbers exactly. For instance, $R(5)$ is not known!

It is not that easy to find lower bounds for $K(n)$ either. Erdös proved that $R(k) \geq 2^{k/2}$ $\forall k \geq 3$ using random graphs.

Random graphs

Random graphs were first considered by Erdős. He defined a probability measure on the set of simple graphs on n vertices, so that a random graph is a graph chosen at random with respect to this probability.
He used random graphs to show the existence of graphs with a given property: instead of explicitly constructing one such graph, he shows that the probability that a graph has the required property is >0, which implies the existence of a graph as wanted. This is the so-called <u>probabilistic method</u>.

The probability measure defined by Erdős is the following. Fix $n\in\mathbb{N}$ and $p\in(0,1)$.
Set $V:=\{1,\dots,n\}$. Then

$\mathcal{G}(n,p):=\{G$ simple graphs with $V(G)=V\}$.

$\mathbb{P}_{n,p}$ probability measure given by $\mathbb{P}_{n,p}(G)=p^m(1-p)^{\binom{n}{2}-m}$, where $m=|E(G)|$

<u>Idea</u>: each edge occurs independently with probability p. Indeed:

<u>Lemma</u>

$E_0\subseteq[V]^{(2)}\Rightarrow\mathbb{P}_{n,p}(E(G)\supseteq E_0)=p^{|E_0|}$ and $\mathbb{P}_{n,p}(E(G)\subseteq[V]^2\setminus E_0)=(1-p)^{|E_0|}$.

<u>Proof</u>.

We prove the case $E(G)\supseteq E_0$. Let $m:=|E_0|$

$$\mathbb{P}_{n,p}(E(G)\supseteq E_0)=\sum_{\substack{G\in\mathcal{G}_{n,p}:\\E(G)\supseteq E_0}}\mathbb{P}_{n,p}(G)=\sum_{i=0}^{\binom{n}{2}-m}\ \sum_{\substack{G:\,E(G)\supseteq E_0\\\text{and}\\|E\setminus E_0|=i}}\mathbb{P}_{n,p}(G).$$

If $|E\setminus E_0|=i$ and $E\supseteq E_0$, $|E|=i+m\Rightarrow\mathbb{P}_{n,p}(G)=p^{i+m}(1-p)^{\binom{n}{2}-i-m}$

Moreover $|\{E\supseteq E_0\mid |E\setminus E_0|=i\}|\underset{\uparrow}{=}\binom{\binom{n}{2}-m}{i}$

I can choose i elts in $[V]^2\setminus E_0$ and $|[V]^2\setminus E_0|=\binom{n}{2}-m$

Let $N:=\binom{n}{2}-m\Rightarrow\mathbb{P}_{n,p}(E(G)\supseteq E_0)=\sum_{i=0}^{N}\binom{N}{i}p^{i+m}(1-p)^{N-i}=p^m\sum_{i=0}^{N}\binom{N}{i}p^i(1-p)^{N-i}=$

$$=p^m(p+1-p)^N=p^m.$$

$\boxed{\text{Ex. 13 Prove the case } E(G)\subseteq[V]^2\setminus E_0}$

□

With this lemma we can prove results as the next one easily:

<u>Lemma</u>

$\forall n\geq k\geq 2$, $\mathbb{P}_{n,p}(G\text{ has }k\text{ indep. vertices})\leq\binom{n}{k}(1-p)^{\binom{k}{2}}$

$\mathbb{P}_{n,p}(G\supseteq K_k)\leq\binom{n}{k}p^{\binom{k}{2}}$.

<u>Proof</u>: A fixed set $U\subseteq V$ is independent in G iff $E\subseteq[V]^2\setminus[U]^2$, so

$$\mathbb{P}_{n,p}(U\text{ is an indep. set})=\mathbb{P}_{n,p}(E\subseteq[V]^2\setminus[U]^2)=(1-p)^{|[U]^2|}=(1-p)^{\binom{|U|}{2}}$$

Since $\{G\mid G\text{ has }k\text{ indep. vertices}\}=\bigcup_{\substack{U\subseteq V,\\|U|=k}}\{G\mid U\text{ is indep. in }G\}$, we have

$|\{U\subseteq V\mid |U|=k\}|=\binom{n}{k}$

$$\mathbb{P}_{n,p}(G\text{ has }k\text{ indep. vertices})\leq\sum_{\substack{U\subseteq V,\\|U|=k}}\mathbb{P}_{n,p}(U\text{ indep. set})=\binom{n}{k}(1-p)^{\binom{k}{2}}.$$

The case $G \supseteq K_r$ is analogous.

$\square$

We want to use this result to prove:

Thm (Erdős)

$\forall k \geq 3$, $R(k) > 2^{k/2}$.

Proof: $R(3) \geq 3 > 2^{3/2}$, so let's assume $k \geq 4$. We need to show that $\forall n \leq 2^{k/2}$, $\exists G : |V| = n$ s.t. G doesn't contain any K_k nor any set of k indep. vertices.

Fix $p = \frac{1}{2}$; we drop p from the notation for the proof.

Claim: $\forall n \leq 2^{k/2}$, $\mathbb{P}_n(G \text{ has } k \text{ indep. vertices}) < \frac{1}{2}$, $\mathbb{P}_n(G \text{ contains a } K_k) < \frac{1}{2}$.

If we prove the claim, then

$$\mathbb{P}_n(G \text{ has no set of } k \text{ indep. vertices \& contains no } K_k) =$$

$$= 1 - \mathbb{P}_n(G \supseteq k \text{ indep. vertices or a } K_r) \geq 1 - (\underbrace{\mathbb{P}_n(G \supseteq k \text{ indep. vertices})}_{<\frac{1}{2}} + \underbrace{\mathbb{P}_n(G \text{ contains a } K_k)}_{<1/2}) > 0$$

$\Rightarrow \exists G$ as required.

So let's prove the claim: by the previous lemma,

$$\mathbb{P}_n(G \text{ contains a } K_k), \mathbb{P}_n(G \text{ contains } k \text{ indep. vertices}) \leq \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} =$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{1}{2^{k(k-1)/2}} < \frac{n^k}{2^k} \frac{1}{2^{k(k-1)/2}} \leq$$

$n(n-1)\ldots(n-k+1) < n^k$ &

$k! > 2^k$ for any $k \geq 4$ (can be shown by induction:

$k = 4 \Rightarrow 4! = 24, 2^4 = 16$

$k \rightsquigarrow k+1 \Rightarrow (k+1)! = (k+1)k! >$
$(k+1)2^k > 2 \cdot 2^k = 2^{k+1}$)

$n \leq 2^{k/2}$

$$\leq \frac{2^{k^2/2}}{2^{k(k+1)/2}} = 2^{-k/2} < \frac{1}{2}.$$

$\square$