Lemma

G connected, finite graph. Then $|E(G)| \geq |V(G)| - 1$ with equality iff G is a tree.

Proof. By induction on $|V(G)|$.

- $|V(G)| = 1$: $|E(G)| \geq 0 = 1-1$; moreover $|E(G)| \neq 0 \iff \exists$ loop $\iff G$ not a tree.
- Supp. true $\forall\, G'$ w/ $|V(G')| \leq |V(G)| - 1$, $|V(G)| \geq 2$.

Since $|V(G)| \geq 2$ and G is connected, there is an edge e joining distinct vertices v, w. Let G' be the graph obtained by collapsing all edges between v and w:

$|V(G')| = |V(G)| - 1$ and $|E(G')| \leq |E(G)| - 1$. Moreover, G' is connected: given any two vertices in G', lift them to G. Then we can find a path between the lifts in G and project it down to a path in G'.

So G' satisfies the induction hypothesis, so

$$|E(G)| - 1 \geq |E(G')| \geq |V(G')| - 1 = |V(G)| - 2 \quad (\star)$$

i.e. $|E(G)| \geq |V(G)| - 1$. Furthermore, $|E(G)| = |V(G)| - 1$ iff all inequalities in $(\star)$ are equalities, i.e.

- there is a unique edge between v and w in G
- $|V(G')| - 1 = |E(G')|$, i.e. G' is a tree.

If G is not a tree, there is a nontrivial cycle in G. As there is a unique edge e in G between v and w, the cycle contains at most one edge between v and w, so it projects to a nontrivial cycle in G', a contradiction.

Conversely: if G is a tree, there can be at most one edge between any two vertices, so $\exists!$ edge between v and w. Moreover, G' is a tree, otherwise it contains a non-trivial cycle, that can be lifted to G $\lightning$.

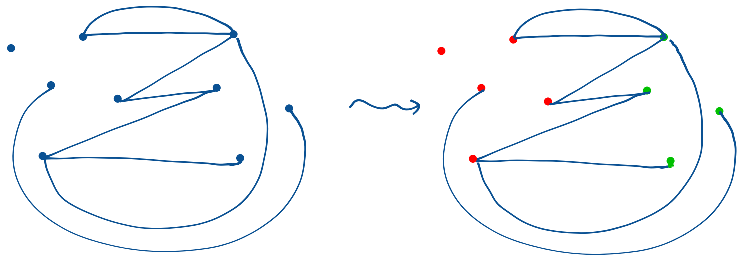
$\square$

A graph G is bipartite if we can write $V(G)$ as $L \sqcup R$ so that $\forall e \in E(G)$ joins a vertex in L to a vertex in R. We call $\{L, R\}$ a bipartition of G.

This means that there are no edges with both endpoints in L, nor with both endpts in R.

We can think of bipartite graphs as graphs with vertices colored red or green and edges only joining vertices of different colors.

Ex.

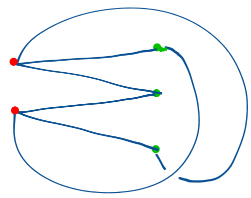


Ex. K_n: $n=2 \Rightarrow$ bipartite

$n \geq 3 \Rightarrow$ not bipartite: if we split $V = V_1 \sqcup V_2$, $V_i \neq \emptyset \Rightarrow$ at least one V_i has size $2 \Rightarrow \exists$ edge between the two vertices $\lightning$

Ex. complete bipartite graphs $K_{n,m}$, $n,m \geq 1$: $V(K_{n,m}) = L \sqcup R$, $|L| = n$, $|R| = m$ and there is 1! edge between any two vertices $v \in L$ and $w \in R$

$n=2$, $m=3$



Rmk: if a bipartition exists, it isn't always unique. For instances, vertices of degree zero (isolated vertices) can belong to any set of a bipartition.

Prop.
G graph. G bipartite $\Leftrightarrow$ G contains no closed walk of odd length.

Proof:
$\boxed{\Rightarrow}$ Let $\{L, R\}$ be a bipartition and $\gamma = (v_0, e_1, \dots, v_n)$ a closed walk.
Without loss of generality, assume $v_0 \in L$. As v_0 and v_1 are adjacent, $v_1 \in R$.
But then $v_2 \in L$, $v_3 \in R$ ... i.e. $v_i \in L$ if i even, $v_i \in R$ if i odd.
$v_n = v_0 \in L \Rightarrow n$ even, and n is the length of γ.

$\boxed{\Leftarrow}$ We use the hypothesis to construct a bipartition. Assume G is connected.
Pick $v \in V$. Define $L := \{w \in V \mid \exists$ walk from v to w of even length$\}$
$R := \{w \in V \mid \exists$ walk from v to w of odd length$\}$.

Claim 1: $L \sqcup R = V$.

Indeed, since G is connected, $\forall w\ \exists$ walk from v to $w \Rightarrow w \in L \cup R \Rightarrow L \cup R = V$.
Moreover, if by contradiction $L \cap R \neq \emptyset$, let $w \in L \cap R$. Then $\exists\ \gamma_1, \gamma_2$ walks from v to w, $\ell(\gamma_1)$ even and $\ell(\gamma_2)$ odd

$\Rightarrow \delta := \overline{\gamma_2} * \gamma_1$ closed walk and $\ell(\delta) = \ell(\gamma_1) + \ell(\gamma_2)$ odd $\lightning$

<u>Claim 2</u>: if e is an edge, it joins a vertex in L to a vertex in R.

Indeed, let $b(e)=\{v_1,v_2\}$. Wlog, $v_1 \in L \Rightarrow \exists\, \gamma$ walk from v to v_1 of even length.
$\delta := (v_1, e, v_2) \Rightarrow \delta * \gamma$ is a walk from v to v_2 of length $\ell(\gamma)+1 \Rightarrow$ odd $\Rightarrow v_2 \in R$.

By the two claims, $\{L, R\}$ is a bipartition.

<u>Ex. 9</u> (1) Show that G is bipartite iff each conn. comp. is bipartite.
(2) Use (1) to conclude the proof (for G not nec. connected).

$\square$

Adjacency matrix and Laplacian

G finite graph. Say $V=\{v_1,\dots,v_n\}$. The <u>adjacency matrix</u> of G is the $n\times n$ matrix $A=(A_{ij})_{i,j=1,\dots,n}$, where

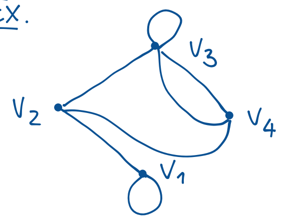
$$A_{ij} := \begin{cases} \#\text{ edges between } v_i \text{ and } v_j, & i\neq j \\ 2\#\text{loops based at } v_i, & i=j \end{cases}$$

It can be useful to think of replacing each edge of G with 2 directed edges with opposite orientation; then $A_{ij} = \#$ <u>directed</u> edges from v_i to v_j.

Note that $\forall i$

$$\deg(v_i) = \left|\{e \text{ not loop} \mid v_i\in b(e)\}\right| + 2\left|\{e \mid b(e)=\{v_i\}\}\right| = \sum_{j\neq i} \left|\{e \mid b(e)=\{v_i,v_j\}\}\right| + A_{ii} =$$
$$= \sum_{j\neq i} A_{ij} + A_{ii} = \sum_{j=1}^{n} A_{ij}.$$

<u>Ex.</u>



$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

<u>Rmk</u>: we could have chosen a different ordering for the vertices! Luckily, this doesn't change things that much:

<u>Lemma</u>

G finite graph, $V=\{v_1,\dots,v_n\} = \{w_1,\dots,w_n\}$, where $w_i = v_{\sigma(i)}\ \forall i$, for some $\sigma\in S_n$. If A is the adj matrix obtained using $\{v_1,\dots,v_n\}$ and B the one obtained using $\{w_1,\dots,w_n\}$, then $A = P_\sigma^t B P_\sigma$, where P_σ is the permutation matrix of σ.

<u>Recall</u>: $\sigma\in S_n \Rightarrow P_\sigma = (p_{ij})_{i,j}$, where $p_{ij} = \delta_{\sigma(i)j} = \begin{cases} 0 & \text{if } j\neq\sigma(i) \\ 1 & \text{if } j=\sigma(i)\end{cases}$

<u>Proof</u>: $$(P_\sigma^t B P_\sigma)_{ij} = \sum_{k=1}^{n} (P_\sigma^t)_{ik} (BP_\sigma)_{kj} = \sum_{k=1}^{n} p_{ki} \sum_{\ell=1}^{n} B_{k\ell}\, p_{\ell j} = \sum_{k,\ell} \underbrace{\delta_{\sigma(k)i} B_{k\ell}\, \delta_{\sigma(\ell)j}}_{\neq 0 \text{ iff } \sigma(k)=i,\ \sigma(\ell)=j} =$$

$$= B_{\sigma^{-1}(i)\sigma^{-1}(j)} = \begin{cases} \#\text{ edges } \overset{v_i}{\overset{\parallel}{\boxed{w_{\sigma^{-1}(i)}}}} \text{ to } \overset{v_j}{\overset{\parallel}{\boxed{w_{\sigma^{-1}(j)}}}}, & i\neq j \\ 2\#\text{ loops based at } \underset{v_i}{\underset{\parallel}{\boxed{w_{\sigma^{-1}(i)}}}}, & i=j \end{cases} = A_{ij}$$

$\square$

As (most) properties of the adjacency matrix we will consider are invariant under matrix congruence, we won't pay attention to the ordering we choose for the vertices. Sometimes we will just write $A=(A_{vw})_{v,w\in V}$.

<u>Ex.</u> The adjacency matrix of K_n is $\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$ (with respect to any ordering of V)

<u>Ex. 10</u> (1) Compute the adjacency matrix of the n-cycle C_n.

(2) Draw the graph whose adjacency matrix is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$.

The reason why we care about adjacency matrices is that they encode many properties of the associated graph. The first connection is with the number of walks between vertices:

Lemma Suppose G has no loops.
$\forall \ell \geq 1$, $(A^\ell)_{ij} = \#$ walks from v_i to v_j of length ℓ.

Proof. Induction on ℓ:
$\ell = 1$: by definition (a walk of length 1 corresponds to an edge)
$\ell \to \ell+1$: $(A^{\ell+1})_{ij} = \sum_{k=1}^{n} \underbrace{(A^\ell)_{ik}}_{\substack{= \\ \#\text{walks} \\ v_i \to v_k \\ \text{of length } \ell}} \underbrace{A_{jk}}_{\substack{= \\ \#\text{edges} \\ v_k \to v_j}} = \sum_{k=1}^{n}$ # walks $v_i \rightsquigarrow v_j$ of length $\ell+1$ w/ before-last vertex v_k $=$ # walks of length $\ell+1$ from v_i to v_j.

(# walks $v_i \to v_k$ of length $\ell+1$ w/ before-last vertex v_j)

$\square$

> **Ex. 11** Prove that $tr(A^\ell) = 0$ iff there are no closed walks of length ℓ. (G without loops.)

Note that the adjacency matrix is a real symmetric matrix, so it has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and there exist an associated orthonormal basis of eigenvectors. $\lambda_1 \geq \dots \geq \lambda_n$ is the spectrum of the matrix, denoted spec(A).

Ex. • $K_2 \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\chi_A(t) = \det(t\,Id - A) = \begin{vmatrix} t & -1 \\ -1 & t \end{vmatrix} = t^2 - 1 = (t+1)(t-1)$

$\lambda_1 = 1 > \lambda_2 = -1$

• $K_3 \Rightarrow A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \chi_A(t) = \begin{vmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} = t^3 - 2 - 3t = (t-2)(t+1)^2$

$\lambda_1 = 2 > \lambda_2 = -1 = \lambda_3$

We want to relate properties of spec(A) with properties of G. For this we need a characterization of eigenvalues using Rayleigh quotients.

Recall: the Rayleigh quotient for A and $f \neq 0$ is

$$R(A,f) := \frac{\langle Af, f\rangle}{\langle f, f\rangle}.$$

We have:

- $R(A,f) \leq \lambda_1 \; \forall f \neq 0$, with equality iff f is an eigenvector for λ_1. In particular, $\max\limits_{f\neq 0} R(A,f) = \lambda_1$.
- if f_1 is an eigenvector for λ_1, $\lambda_2 = \max\limits_{\substack{f \perp f_1,\\ f \neq 0}} R(A,f)$.

Indeed: if $f_1, \dots, f_n$ is an orthonormal basis of eigenvectors for $\lambda_1, \dots, \lambda_n$ and $f \neq 0$, then we can write

$$f = \sum_{i=1}^n x_i f_i$$

and if $f \neq 0$, $R(A,f) = \frac{\langle Af, f\rangle}{\langle f, f\rangle} = \frac{\sum \lambda_i x_i^2}{\sum x_i^2} \underset{\substack{\uparrow\\ \lambda_i \leq \lambda_1}}{\leq} \frac{\lambda_1 \sum x_i^2}{\sum x_i^2} = \lambda_1.$

If $Af = \lambda_1 f \Rightarrow R(A,f) = \frac{\langle \lambda_1 f, f\rangle}{\langle f, f\rangle} = \lambda_1$. If $R(A,f) = \lambda_1 \Rightarrow \lambda_i = \lambda_1 \; \forall i : x_i \neq 0$, i.e. f eigenvector for λ

If $f \perp f_1 \Rightarrow f = \sum_{i=2}^n x_i f_i \Rightarrow R(A,f) = \frac{\sum_{i=2}^n \lambda_i x_i^2}{\sum_{i=2}^n x_i^2} \leq \lambda_2$, so $\max\limits_{\substack{f \perp f_1\\ f \neq 0}} R(A,f) \leq \lambda_2$. Since $R(A,f_2) = \lambda_2$,

$$\max_{\substack{f \perp f_1\\ f \neq 0}} R(A,f) = \lambda_2.$$

Prop.
G finite graph with spectrum $\lambda_1 \geq \dots \geq \lambda_n$.
(1) $\exists f$ eigenvector for λ_1 with $f_v \geq 0 \; \forall v \in V$.
(2) Supp. G connected. Then:

- $\exists f$ eigenvector for λ_1 w/ $f_v > 0 \; \forall v \in V$.
- if $g \neq 0$ is an eigenvector with $g_v \geq 0 \; \forall v$, then $Ag = \lambda_1 g$ and $g_v > 0 \; \forall v$.

Proof.
(1) Let f be an eigenvector for $\lambda_1 \Rightarrow R(A,f) = \lambda_1$.

$\bar{f} := (\bar{f}_v = |f_v|)_{v \in V}$; then

$$\langle \bar f, \bar f\rangle = \sum \bar{f}_v^2 = \sum f_v^2 = \langle f, f\rangle \text{ and } \langle A\bar f, \bar f\rangle = \sum_{v,w} \underbrace{A_{vw}}_{\geq 0} \underbrace{\bar f_v \bar f_w}_{\geq f_v f_w} \geq \sum A_{vw} f_v f_w = \langle Af, f\rangle$$

$\Rightarrow \lambda_1 \geq \frac{\langle A\bar f, \bar f\rangle}{\langle \bar f, \bar f\rangle} \geq \frac{\langle Af, f\rangle}{\langle f, f\rangle} \geq \lambda_1 \Rightarrow \bar f$ eigenvector for λ_1 w/ non-neg. entries.

(2) By (1), $\exists f \neq 0$ eigenvect. for λ_1 with non-negative entries. Claim: $f_v > 0 \; \forall v$.

If not, let $S := \{v \in V \mid f_v = 0\}$; then $S \neq \emptyset$ and (since $f \neq 0$) $S \neq V$.

$\forall v \in S$: $0 = \lambda_1 f_v = (Af)_v = \sum_{w \in V} A_{vw} f_w \underset{\substack{\uparrow\\ f_w = 0 \text{ if } w \in S}}{=} \sum_{w \notin S} \underbrace{A_{vw}}_{\geq 0} \underbrace{f_w}_{> 0} \Rightarrow A_{vw} = 0 \; \forall w \notin S$

$\Rightarrow$ there are no edges from S to $V \setminus S$, i.e. G is disconnected $\lightning$ So $f_v > 0 \; \forall v$.

Now suppose $g \neq 0$, $g_v \geq 0\ \forall v$ and $Ag = \lambda g$. If $\lambda \neq \lambda_1$, $g \perp f\ \forall f$ ev of eigenvalue λ_1.
Choose f s.t. $f_v > 0\ \forall v$; then

$$0 = \langle f, g \rangle = \underbrace{\sum f_v g_v}_{\geq 0} \Rightarrow f_v g_v = 0\ \forall v \underset{f_v > 0}{\Rightarrow} g_v = 0\ \forall v \ \lightning$$

As before then $g_v > 0\ \forall v$. □

Prop.
G finite. Then G connected $\Rightarrow \lambda_1 < \lambda_2$ (i.e. λ_1 has multiplicity 1).

Proof. By the previous proposition, $\exists f$ ev for λ_1 with $f_v > 0\ \forall v$.
Suppose by contradiction that λ_1 has multiplicity at least 2. Let $f' \neq 0$ be an eigenvector for λ_1 with $f' \perp f$.

$$\Downarrow$$

$$\sum_{v \in V} f'_v \underbrace{\boxed{f_v}}_{>0\ \forall v} = 0 \Rightarrow f'_v \text{ has positive and negative entries.}$$

So $\exists \alpha \in \mathbb{R}$ s.t. $\min_{v \in V}(f'_v - \alpha f_v) = 0$.

Set $\bar{f} := f' - \alpha f$; since $f' \perp f$, $\bar{f} \neq 0$, and it's an eigenvector for λ_1. Moreover $\bar{f}_v \geq 0\ \forall v$, so by the prop. $\bar{f}_v > 0\ \forall v$, but by construction $\exists v : \bar{f}_v = 0$ $\lightning$

So $\lambda_2 > \lambda_1$.

□