Adjacency matrix and Laplacian

G finite graph. Say $V=\{v_1,\dots,v_n\}$. The <u>adjacency matrix</u> of G is the $n\times n$ matrix $A=(A_{ij})_{i,j=1,\dots,n}$, where

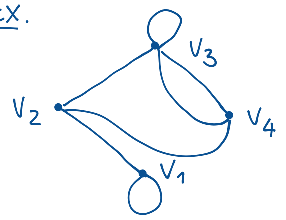
$$A_{ij} := \begin{cases} \#\text{ edges between } v_i \text{ and } v_j, & i\neq j \\ 2\#\text{loops based at } v_i, & i=j \end{cases}$$

It can be useful to think of replacing each edge of G with 2 directed edges with opposite orientation; then $A_{ij} = \#$ <u>directed</u> edges from v_i to v_j.

Note that $\forall i$

$$\deg(v_i) = \left|\{e \text{ not loop} \mid v_i\in b(e)\}\right| + 2\left|\{e \mid b(e)=\{v_i\}\}\right| = \sum_{j\neq i}\left|\{e \mid b(e)=\{v_i,v_j\}\}\right| + A_{ii} =$$
$$= \sum_{j\neq i} A_{ij} + A_{ii} = \sum_{j=1}^{n} A_{ij}.$$

<u>Ex.</u>



$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

<u>Rmk</u>: we could have chosen a different ordering for the vertices! Luckily, this doesn't change things that much:

<u>Lemma</u>

G finite graph, $V=\{v_1,\dots,v_n\}=\{w_1,\dots,w_n\}$, where $w_i = v_{\sigma(i)}\ \forall i$, for some $\sigma\in S_n$. If A is the adj matrix obtained using $\{v_1,\dots,v_n\}$ and B the one obtained using $\{w_1,\dots,w_n\}$, then $A = P_\sigma^t B P_\sigma$, where P_σ is the permutation matrix of σ.

<u>Recall</u>: $\sigma\in S_n \Rightarrow P_\sigma = (p_{ij})_{i,j}$, where $p_{ij} = \delta_{\sigma(i)j} = \begin{cases} 0 & \text{if } j\neq\sigma(i) \\ 1 & \text{if } j=\sigma(i)\end{cases}$

<u>Proof</u>: $$(P_\sigma^t B P_\sigma)_{ij} = \sum_{k=1}^{n}(P_\sigma^t)_{ik}(BP_\sigma)_{kj} = \sum_{k=1}^{n} p_{ki}\sum_{l=1}^{n} B_{kl}\,p_{lj} = \sum_{k,l}\underbrace{\delta_{\sigma(k)i}\,B_{kl}\,\delta_{\sigma(l)j}}_{\neq 0 \text{ iff } \sigma(k)=i,\ \sigma(l)=j} =$$

$$= B_{\sigma^{-1}(i)\sigma^{-1}(j)} = \begin{cases} \#\text{ edges } \overset{v_i}{\overset{\parallel}{\boxed{w_{\sigma^{-1}(i)}}}} \text{ to } \overset{v_j}{\overset{\parallel}{\boxed{w_{\sigma^{-1}(j)}}}}, & i\neq j \\ 2\#\text{ loops based at } \underset{v_i}{\underset{\parallel}{\boxed{w_{\sigma^{-1}(i)}}}}, & i=j \end{cases} = A_{ij}$$

$\square$

As (most) properties of the adjacency matrix we will consider are invariant under matrix congruence, we won't pay attention to the ordering we choose for the vertices. Sometimes we will just write $A=(A_{vw})_{v,w\in V}$.

<u>Ex.</u> The adjacency matrix of K_n is $\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$ (with respect to any ordering of V)

<u>Ex. 9</u> (1) Compute the adjacency matrix of the n-cycle C_n.

(2) Draw the graph whose adjacency matrix is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$.

The reason why we care about adjacency matrices is that they encode many properties of the associated graph. The first connection is with the number of walks between vertices:

Lemma
$\forall \ell \geq 1, (A^\ell)_{ij} = \#$ (oriented) walks from v_i to v_j of length ℓ.

Proof. Induction on ℓ:
$\ell = 1$: by definition (a directed walk of length 1 is a directed edge).
$\ell \to \ell+1$: $$(A^{\ell+1})_{ij} = \sum_{k=1}^{n} \underbrace{\underbrace{(A^\ell)_{ik}}_{\substack{\#\text{ dir. walks}\\ v_i \to v_k\\ \text{of length } \ell}} \underbrace{A_{jk}}_{\substack{\#\text{ dir. edges}\\ v_k \to v_j}}}_{\substack{\#\text{ dir. walks } v_i \to v_k\\ \text{of length } \ell+1 \text{ w/}\\ \text{before-last vertex } v_j}} = \sum_{k=1}^{n} \#\text{ or. walks } v_i \rightsquigarrow v_j \text{ of length } \ell+1 \text{ w/ before-last vertex } v_k = \#\text{ or. walks of length } \ell+1 \text{ from } v_i \text{ to } v_j.$$

□

Ex. 10 Prove that $\mathrm{tr}(A^\ell) = 0$ iff there are no closed walks of length ℓ.

Note that the adjacency matrix is a real symmetric matrix, so it has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and there exist an associated orthonormal basis of eigenvectors. $\lambda_1 \geq \dots \geq \lambda_n$ is the spectrum of the matrix, denoted $\mathrm{spec}(A)$.

Ex. • $K_2 \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\chi_A(t) = \det(tId - A) = \begin{vmatrix} t & -1 \\ -1 & t \end{vmatrix} = t^2 - 1 = (t+1)(t-1)$

$\lambda_1 = 1 > \lambda_2 = -1$

• $K_3 \Rightarrow A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \chi_A(t) = \begin{vmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} = t^3 - 2 - 3t = (t-2)(t+1)^2$

$\lambda_1 = 2 > \lambda_2 = -1 = \lambda_3$

Prop.
G finite, d-regular.
(1) $\mathrm{spec}(A) \subseteq [-d, d]$, $\lambda_1 = d$ and $(1, \dots, 1)$ is an ev for λ_1.
(2) G connected $\Leftrightarrow \lambda_2 < d$.
(3) The following are equivalent:
 (a) G bipartite;
 (b) $\mathrm{spec}(A) = -\mathrm{spec}(A)$ (i.e. if $\alpha \in \mathrm{spec}(A)$, $-\alpha \in \mathrm{spec}(A)$ and α and $-\alpha$ have the same mult.)
 (c) $\forall \ell$ odd, $\mathrm{tr}(A^\ell) = 0$.

Proof:
(1) $\lambda \in \mathrm{spec}(A)$ with eigenvector f; $\lambda f = Af = \left(\sum_u A_{vu} f_u\right)_v$.
Let w be such that $|f_w| = \max_{u \in V} |f_u|$. Then

$$|\lambda||f_w| = \Big|\sum_{u\in V} A_{wu} f_u\Big| \underset{\substack{\uparrow\\ A_{wu} \geq 0}}{\leq} \sum_{u \in V} A_{wu} \underbrace{|f_u|}_{\substack{\wedge\!\!|\\ |f_w|}} \leq |f_u| \sum_{u\in V} A_{wu} = d|f_u| \Rightarrow |\lambda| \leq d.$$

If we show that d is an eigenvalue, it's automatically λ_1 by (1). We just check that $A(1, \dots, 1) = d(1, \dots, 1)$. Indeed: $A(1, \dots, 1) = \left(\sum_{u \in V} A_{uv}\right)_v = (d)_v = d(1)_v$.

(2) $\boxed{\Leftarrow}$ Supp. by contradiction G is not connected. Let S be the set of vertices of a conn. component $\Rightarrow \forall v \in S, w \in \bar{S}$, there is no edge from v to w, i.e. $A_{vw} = 0$

$\Rightarrow A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B is the adjacency matrix of the conn. comp. given by S and C the adj matrix given by all other conn. components.

Rmk: G d-reg. $\Rightarrow$ its conn. comp. are d-regular $\Rightarrow B\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = d\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

$\Rightarrow A \underbrace{\begin{pmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}}_{\overset{!!}{f}} = \begin{pmatrix} B\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \\ C\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\end{pmatrix} = \begin{pmatrix} d\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \\ \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\end{pmatrix} = d\begin{pmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} \Rightarrow f$ ev for d, $f \notin \left\langle \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \right\rangle \Rightarrow d$ has multiplicity ≥ 2, i.e. $\lambda_2 = d$ $\lightning$

$\boxed{\Rightarrow}$ By contradiction, supp. $\lambda_2 = d \Rightarrow \exists f \notin \left\langle \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \right\rangle$ s.t. $Af = df$. Let $\alpha \in \mathbb{R}$ be s.t.

$$\min_{v \in V} \{f_v + \alpha\} = 0.$$

Set $f' := f + \alpha\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$; note: $f' \notin \left\langle \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \right\rangle$, otherwise $f \in \left\langle \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \right\rangle$. In particular $\exists v: f'_v > 0$.

Moreover f' is an ev for d (since $f' \in \left\langle f, \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \right\rangle$).

Let $S := \{v \in V \mid f'_v > 0\} \neq \phi$.

$\underset{\text{because } \exists w: f'_w = 0}{\neq V}$

Claim: S is disconnected from $\bar{S}$.

Indeed: $\forall w \in \bar{S}$, $f'_w = 0$, so

$$0 = df'_w = (Af')_w = \sum_{v \in V} A_{wv} f'_v \underset{f'_v = 0 \ \forall v \in \bar{S}}{=} \sum_{v \in S} A_{wv} f'_v \underset{A_{wv} \geq 0,\ f'_v > 0\ \forall v \in S}{\Rightarrow} A_{wv} = 0 \ \forall v \in S$$

$\Rightarrow \nexists$ edge from S to $\bar{S}$.

(3) (a) $\Rightarrow$ (b) Let $\{L = \{v_1, \dots, v_k\}, R = \{v_{k+1}, \dots, v_n\}\}$ be a bipartition. Then

$$A = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

where $B_{ij} = \#$edges from v_i to v_{k+j} $\Rightarrow B$ is a $k \times (n-k)$ matrix

Supp. $\alpha \in \operatorname{spec}(A)$ and let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be an eigenvector for α, where f_1 has $(n-k)$ entries and f_2 k entries.

Then $Af = \alpha f \Leftrightarrow \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \alpha \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \Leftrightarrow \begin{cases} Bf_2 = \alpha f_1 \\ B^t f_1 = \alpha f_2 \end{cases}$

Let $f' := \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} \Rightarrow Af' = \begin{pmatrix} -Bf_2 \\ B^t f_1 \end{pmatrix} = \begin{pmatrix} -\alpha f_1 \\ \alpha f_2 \end{pmatrix} = -\alpha \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} \Rightarrow -\alpha \in \operatorname{spec}(A)$ and f' is an eigenvector for $-\alpha$. Moreover, $f^{(1)} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \end{pmatrix}, \dots, f^{(m)} = \begin{pmatrix} f_1^{(m)} \\ f_2^{(m)} \end{pmatrix}$ are indep. eigenvectors for α

$\Leftrightarrow \begin{pmatrix} f_1^{(1)} \\ -f_2^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} f_1^{(m)} \\ -f_2^{(m)} \end{pmatrix}$ are indep. eigenvectors of $-\alpha \Rightarrow \alpha$ and $-\alpha$ have the same mult.

(b) ⇒ (c) The trace of A^ℓ is the sum of its eigenvalues, which are the ℓ-th power of the eigenvalues of A, so

$$\mathrm{tr}(A^\ell) = \sum_{\alpha\in\mathrm{spec}(A)} \alpha^\ell \underset{\mathrm{spec}(A)=-\mathrm{spec}(A)}{\uparrow}{=} \frac{1}{2}\Big(\sum_{\alpha\in\mathrm{spec}(A)}\alpha^\ell + \sum_{\alpha\in-\mathrm{spec}(A)}\alpha^\ell\Big) = \frac{1}{2}\sum_{\alpha\in\mathrm{spec}(A)}(\alpha^\ell + (-\alpha)^\ell) \underset{\ell \text{ odd}}{\uparrow}{=} 0$$

(c) ⇒ (a) By ex. 9, there are no closed walks of odd length $\underset{\text{Prop.}}{\Rightarrow}$ G bipartite

□

The <u>Laplacian</u> of a graph G is the matrix

$$L = D - A$$

where $D = \mathrm{diag}(\deg(v_1), \dots, \deg(v_n))$.

Again, L is real and symmetric, so it has real eigenvalues $\mu_n \geq \dots \geq \mu_1$.

<u>Ex. 11</u> Prove that v_i is an isolated vertex (i.e. there are no edges between v_i and $v_j \neq v_i$) $\Leftrightarrow L_{ii} = 0$.

<u>Rmk</u>: G d-regular $\Rightarrow L = d\,\mathrm{Id} - A \Rightarrow$ easy to go from $\mathrm{spec}(A)$ to $\mathrm{spec}(L)$.

To $e \in E$, e not a loop, we associate the $n\times n$ matrix L_e given by:

$$(L_e)_{ij} = \begin{cases} 1 & \text{if } i=j \text{ and } v_i \in b(e) \\ -1 & \text{if } b(e) = \{v_i, v_j\} \\ 0 & \text{otherwise} \end{cases}$$

<u>Lemma</u>

$$L = \sum_{\substack{e \text{ not} \\ \text{loop}}} L_e.$$

<u>Proof</u>:

$$\Big(\sum_{e\in E} L_e\Big)_{ii} = \sum_{\substack{e \text{ not} \\ \text{loop}, \\ v_i \in b(e)}} 1 = \#\{e\in E \mid b(e)\ni v_i, e \text{ not loop}\} = \deg(v_i) - \#\{e\in E \mid b(e_i) = \{v_i\}\} =$$

$$= (D)_{ii} - A_{ii}$$

If $i \neq j$:

$$\Big(\sum_{\substack{e \text{ not} \\ \text{loop}}} L_e\Big)_{ij} = \sum_{\substack{e\in E, \\ b(e)=\{v_i,v_j\}}} (-1) = -\#\{e\in E \mid b(e) = \{v_i, v_j\}\} = -A_{ij} = (D)_{ij} - A_{ij} \quad \text{(since } D \text{ is diagonal)}.$$

□

<u>Rmk</u>: this shows that the Laplacian "doesn't see" loops – if two graphs are the same except for the loops, they have the same Laplacian.

<u>Thm</u>

(1) L is positive semidefinite, i.e. $\forall f \in \mathbb{R}^n$, $\langle f, Lf\rangle \geq 0$. In particular, $\mu_1 \geq 0$.

(2) $\mu_1 = 0$.

(3) The multiplicity of the 0 eigenvalue is the number of connected comp. of G. In particular, G connected $\Leftrightarrow \mu_2 > 0$.

<u>Proof</u>.

(1) Let e be an edge, not a loop, with $b(e) = \{v_{i_0}, v_{i_1}\}$. Then:

$(L_e)_{i_0 i_0} = (L_e)_{i_1 i_1} = 1$, $(L_e)_{i_0 i_1} = (L_e)_{i_1 i_0} = -1$ and the other entries are zero, so

$$\langle f, L_e f\rangle = \sum_{i=1}^n f_i\Big(\sum_{j=1}^n (L_e)_{ij} f_j\Big) = f_{i_0}^2 + f_{i_1}^2 - 2 f_{i_0} f_{i_1} = (f_{i_0} - f_{i_1})^2 \geq 0.$$

So each L_e is positive semidefinite and thus so is L.

If $\mu \in \operatorname{spec}(L)$ and f is a nonzero eigenvector $\Rightarrow 0 \leq \langle f, Lf \rangle = \langle f, \lambda f \rangle = \lambda \underbrace{\langle f, f \rangle}_{\geq 0}$

$$\Rightarrow \lambda \geq \frac{\langle f, Lf \rangle}{\langle f, f \rangle} \geq 0.$$

(2) f vector with all entries 1 $\Rightarrow (Lf)_i = \sum_{J=1}^{n} L_{iJ} f_J = \sum_{J=1}^{n} L_{iJ} = \deg(v_i) - \sum_{J=1}^{n} A_{iJ} = 0$

$$\Rightarrow Lf = 0 \cdot f, \text{ i.e. } \mu_1 = 0.$$

(3) Supp. G has k connected components $C_1, \dots, C_k$.

We will prove that there are k linearly independent eigenvectors for 0 which span the eigenspace of 0. This implies that the multiplicity of 0 is k.

$\forall i$, let $f^{(i)}$ be the vector $\left(f^{(i)}\right)_J = \begin{cases} 1 & \text{if } v_J \in V(C_i) \\ 0 & \text{otherwise} \end{cases}$

Rmk: since $V \quad , \sqcup \dots \sqcup V(C_k) = V$, $\forall J \; \exists!\, i_J$ s.t. $\left(f^{(i_J)}\right)_J \neq 0$

$$\Rightarrow \text{if } \sum_{i=1}^{n} \alpha_i f^{(i)} = 0 \Rightarrow \left(\sum_{i=1}^{n} \alpha_i f^{(i)}\right)_J = 0 \;\forall J \Rightarrow \alpha_J \underbrace{\left(f^{(i_J)}\right)_J}_{1} = 0 \;\forall J \Rightarrow \alpha_J = 0 \;\forall J, \text{ i.e.}$$

$f^{(1)}, \dots, f^{(k)}$ are lin. indep.

Moreover:

$$\left(Lf^{(i)}\right)_J = \sum_{k=1}^{n} (L_{Jk}) \left(f^{(i)}\right)_k = \sum_{\substack{k: \\ v_k \in C_i}} L_{Jk}$$

- if $v_J \notin C_i \Rightarrow J \neq k \;\forall k : v_k \in C_i$ and $L_{Jk} = -A_{Jk} = 0$ since there are no edges between vertices in different components

- if $v_J \in C_i$: $\left(Lf^{(i)}\right)_J = L_{JJ} + \sum_{\substack{k \neq J, \\ v_k \in C_i}} L_{Jk} = \deg(v_J) - \sum_{\substack{k: \\ v_k \in C_i}} A_{Jk} = 0$

$\Rightarrow \left(Lf^{(i)}\right)_J = 0 \;\forall J$, i.e. $Lf^{(i)} = 0$, i.e. $f^{(i)}$ is an eigenvector for 0.

If $f \neq 0$ is such that $Lf = 0$, then

$$0 = \langle f, Lf \rangle = \sum_{e \in E} \langle f, L_e f \rangle = \sum_{i<J} \sum_{\substack{e: \\ b(e) = \{v_J, v_k\}}} (f_J - f_k)^2$$

$\Rightarrow f_J = f_k$ if v_J and v_k are adjacent; by induction, this implies that if v_J and v_k are joined by a walk, $f_J = f_k$.

As each C_i is connected, any two vertices in C_i are joined by some walk, so f_J is the same $\forall J: v_J \in C_i$. Say $f_J = \alpha_i \;\forall J: v_J \in C_i$

$$\Rightarrow f = \sum_{i=1}^{k} \alpha_i f^{(i)}, \text{ i.e. the } f^{(i)} \text{ generate the eigenspace of } 0.$$

Now, G conn. $\Leftrightarrow$ G has $1!$ conn. component $\Leftrightarrow$ the multiplicity of 0 is 1 $\Leftrightarrow \mu_2 > 0$. $\square$