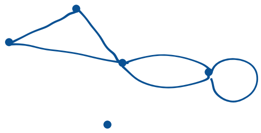
Graphs

Idea: nodes connected by edges



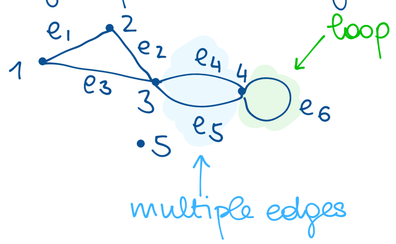
To formalize it:

Graph: a triple $G = (V(G), E(G), b_G)$ where $V(G), E(G)$ sets and $b_G : E(G) \to V(G)^{(2)}$
where $V(G)^{(2)}$ = subsets of $V(G)$ of cardinality 1 or 2

Notation:

Elements of $V(G)$: vertices
Elements of $E(G)$: edges
$v \in b(e)$ is an endpoint of e and e is incident to v.
$v, w \in V(G)$ are adjacent if $\exists e \in E(G) : b_G(e) = \{v, w\}$
$e, f \in E(G)$ are consecutive edges if $b_G(e) \cap b_G(f) \neq \phi$.
Rmk: when the graph G is clear from the context, we drop G from the notation

Ex.1 the graph drawn before is given by

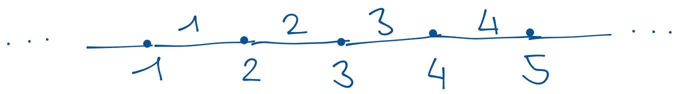


$V = \{1, \dots, 5\}$, $E = \{e_1, \dots, e_6\}$,

$b : e_1 \mapsto \{1,2\}$ $\quad e_4 \mapsto \{3,4\}$
$\quad e_2 \mapsto \{2,3\}$ $\quad e_5 \mapsto \{3,4\}$
$\quad e_3 \mapsto \{1,3\}$ $\quad e_6 \mapsto \{4\}$

Loop $= e \in E(G)$ s.t. $|b_G(e)| = 1$
G has multiple edges if $\exists e \neq e' \in E(G) : b_G(e) = b_G(e')$.

Ex.2 $V = \mathbb{Z}$, $E = \mathbb{Z}$, $b : E \to V^{(2)}$
$\qquad n \mapsto \{n, n+1\}$



G finite if $|V(G)| + |E(G)| < \infty$, infinite otherwise

EX.3 $V = \{*\}$, $E = \mathbb{Z}$, $b : \mathbb{Z} \to \{*\}$
$\qquad n \mapsto \{*\}$

infinite

Ex. $\forall n \geq 3$, the n-cycle C_n is the graph (V, E, b) given by $V = \{1, \dots, n\}$, $E = \{e_1, \dots, e_n\}$
and $b(e_i) = \{i, i+1\}$ for $i < n$, $b(e_n) = \{1, n\}$

G graph, $v \in V$; the degree of v is
$\deg_G(v) = \deg(v) := |\{e \in E \mid v \in b(e), e \text{ not loop}\}| + 2|\{e \in E \mid b(e) = \{v\}\}|$

Idea: replace each edge with two oriented edges with opposite or.; degree = # of or. edges starting at the vertex

In ex 1: $\deg(4) = 4 = \deg(3)$, $\deg(5) = 0$, $\deg(1) = 2$. In ex 2, $\deg(n) = 2 \ \forall n$. In ex. 3, $\deg(*) = \infty$.

Lemma (Handshaking lemma)

G finite graph $(\neq \emptyset)$ $\Rightarrow 2|E(G)| = \sum_{v \in V(G)} \deg(v)$.

Proof. Suppose first G has no loops.

$$\left|\{(v,e) \mid v \in V, e \in E, v \in b(e)\}\right| = \sum_{v \in V(G)} \deg(v)$$

$\| \;=\; 2|E(G)|$ ← each edge has exactly 2 endpts (no loops) $\Rightarrow$ it appears in 2 pairs

$=$ ↑ each v belongs to $\deg(v)$ pairs

If G has loops, let G' be the graph obtained by removing the loops. Then

$$\sum_{v \in V(G')} \deg_{G'}(v) = 2|E(G')|$$

Note that $V(G') = V(G)$, $E(G') = \{e \in E(G) \mid e \text{ not loop}\}$

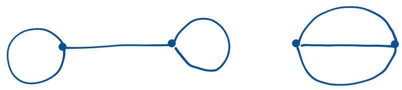
$$\deg_G(v) = \deg_{G'}(v) + 2\left|\{e \in E(G) \mid b_G(e) = \{v\}\}\right|$$

$$\Rightarrow \sum_{v \in V(G)} \deg_G(v) = \sum_{v \in V(G)} \deg_{G'}(v) + 2\sum_{v \in V(G)} \left|\{e \mid b_G(e) = \{v\}\}\right| = 2|E(G')| + 2\left|\{e \mid e \text{ loop}\}\right| = 2|E(G)|.$$

□

G is <u>regular</u> if all vertices have the same degree. If the degree is d, we say G is <u>d-regular</u>.

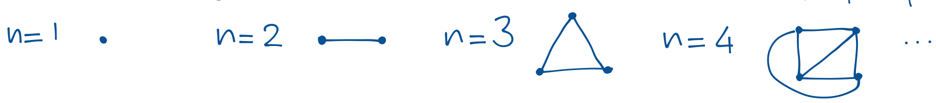
Ex. 3-regular graphs on 2 vertices:



Ex. 1 Show that if G is finite, d-regular, then $|V(G)| = \frac{2}{d}|E(G)|$. Deduce that if d is odd, $|V(G)|$ must be even.

Ex. <u>Complete graphs</u>

$n \geq 1 \Rightarrow K_n$: graph with n vertices and one edge per pair of vertices



K_n is $(n-1)$-regular

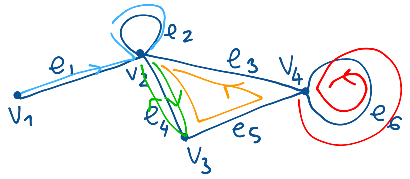
Ex. 2 Find a description of K_n as triple (V, E, b).

Moving on a graph

G graph. A <u>walk</u> on G is an ordered sequence $(v_0, e_1, \ldots, v_n, e_n)$ of vertices v_i and edges e_i s.t. $b(e_i) = \{e_{i-1}, e_i\}$. We say it starts at v_0 and ends at v_n. Its <u>length</u> $\ell(\gamma)$ is n ($= \#$ edges).

Informally: we start at a vertex, follow one edge incident to it to another vertex, then follow another edge and so on. Note that we can pass through the same vertex or the same edge more than once.

Ex.



W_1: $(v_1, e_1, v_2, e_2, v_2)$

W_2: $(v_2, e_4, v_3, e_4, v_2)$

W_3: $(v_2, e_4, v_3, e_5, v_4, e_3, v_2)$

W_4: $(v_4, e_6, v_4, e_6, v_4)$

A walk is closed if it starts and ends at the same vertex.

Ex. w_2, w_3, w_4, but not w_1

A cycle is a closed walk such that all vertices, except the first and the last, are distinct. It is nontrivial if it has positive length. Rmk: essentially a cycle is an embedding of C_n for some n

Ex. w_3. not w_2 nor w_4

We can also concatenate walks, if the first ends where the second one starts, and follow a walk backwards:

$\gamma = (v_0, e_1, \dots, v_n)$, $\delta = (w_0, f_1, \dots, w_m)$ s.t. $v_n = w_0 \Rightarrow$ the concatenation $\delta * \gamma$ is the walk $(v_0, e_1, \dots, v_n = w_0, f_1, \dots, w_m)$.

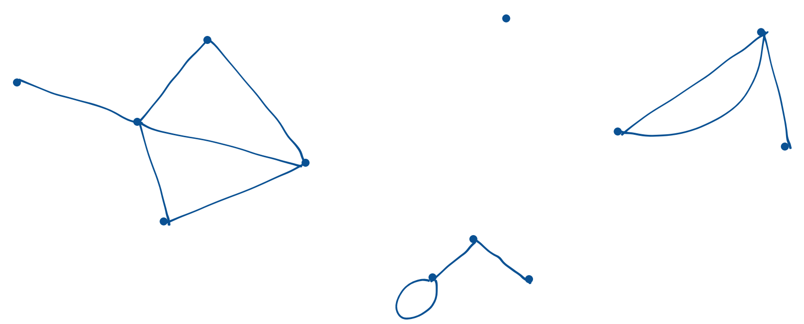
The inverse walk is the walk $\bar{\gamma} = (v_n, e_n, v_{n-1}, \dots, e_1, v_0)$.

We want to distinguish graphs made of "one piece" or "multiple pieces"; for this we use walks: "one piece" means that we can walk between any two vertices.

A graph G is connected if $\forall v, w \in V(G)$ $\exists$ walk from v to w and disconnected otherwise.

Ex. 3 The relation on $V(G)$ given by $v \sim w$ if $\exists$ walk from v to w is an equivalence relation, i.e. it's
(1) reflexive ($\forall v \in V(G)$, $v \sim v$)
(2) symmetric (if $v \sim w$, then $w \sim v$)
(3) transitive (if $v \sim w$ and $w \sim z$, then $v \sim z$).

Ex.



Not connected, made of 4 "pieces". To formalize this, we can use the concept of eq. classes.

$v \in V(G)$; its equivalence class is $[v] := \{w \in V(G) \mid v \sim w\}$.

Note: given v_1, v_2, either $[v_1] = [v_2]$ or $[v_1] \cap [v_2] = \emptyset$: indeed, if $[v_1] \cap [v_2] \neq \emptyset$ then $\exists\, w_0 \in [v_1] \cap [v_2]$, i.e. $v_1 \sim w_0$ and $v_2 \sim w_0$; $\sim$ symm $\Rightarrow w_0 \sim v_1$; $\sim$ trans $\Rightarrow v_2 \sim v_1$
$[v_1] \subseteq [v_2]$ because if $w \in [v_1] \Rightarrow v_1 \sim w \underset{\sim \text{trans}}{\Rightarrow} v_2 \sim w \Rightarrow w \in [v_2]$.

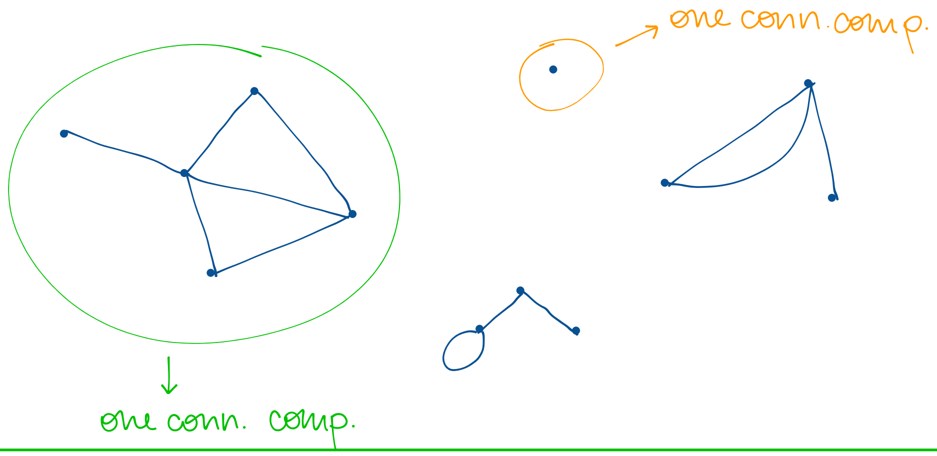
A connected component of a graph G is the graph $G_{[v]}$ (for some $v \in V$) given by:

$$V(G_{[v]}) = [v]$$
$$E(G_{[v]}) = \{e \in E(G) \mid b_G(e) \subseteq [v]\}$$
$$b_{G_{[v]}}(e) = b(e) \quad \forall e \in E(G_{[v]}).$$

$G_{[v]}$: conn. comp. containing v

We are looking at all vertices connected to some vertex (and hence to each other) and all the edges between them.

Ex.



Ex. 4 C_1, C_2 distinct connected components of $G \Rightarrow \nexists$ no edge from $V(C_1)$ to $V(C_2)$

Rmk: G connected $\Leftrightarrow$ it has $\exists!$ connected component.

Walks can also be used to turn graphs into metric spaces.

A metric space is a pair (X, d), where X is a set and a fct $d: X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ s.t.

(1) $d(x,y) = 0 \Leftrightarrow x = y$;

(2) $\forall x, y \in X$, $d(x,y) = d(y,x)$;

(3) $\forall x,y,z \in X$ $d(x,z) \leq d(x,y) + d(y,z)$ (triangular inequality)

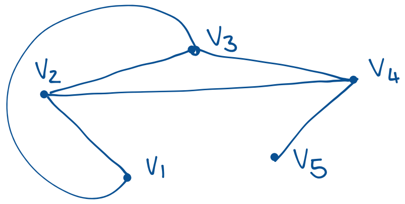
d is called distance.

To a graph G, we associate the metric space (V, d_G), where

$$d_G(v,w) := \inf \{ \ell(\gamma) \mid \gamma \text{ walk from } v \text{ to } w \}$$

is the combinatorial metric. We will just write d when it's clear which graph we are considering.

Ex.



$d(v_1, v_2) = 1$ $\quad d(v_1, v_4) = 2$

$d(v_1, v_5) = 3$

Rmk: if G is not connected, $\exists v, w$ w/o any path from v to $w \Rightarrow d_G(v,w) = \inf \emptyset = \infty$.

Ex. 5 Show that (V, d_G) is a metric space.

The diameter of a graph G is

$$\operatorname{diam}(G) := \sup \{ d(v,w) \mid v, w \in V \}.$$

Lemma

G finite $\Rightarrow \operatorname{diam}(G) < \infty \Leftrightarrow G$ connected.

Proof. $\boxed{\Rightarrow}$ $\forall v, w \in V$, $d(v,w) \leq \operatorname{diam}(G) < \infty \Rightarrow \exists$ walk between them. So G is connected.

$\boxed{\Leftarrow}$ G conn. $\Rightarrow \forall v, w \in V$ $\exists$ walk between them, so $d(v,w) < \infty$. Since G is finite, $\operatorname{diam}(G)$ is the supremum of finitely many integers, so it's finite. $\square$

Ex. Finiteness is fundamental in the previous lemma: $\cdots - \bullet - \bullet - \bullet - \cdots$ is conn, but diam $= \infty$

Ex. 6 Compute $\operatorname{diam}(K_n)$ and $\operatorname{diam}(C_n)$, $n \geq 3$.

Lemma

G with $\max_{v \in V} \deg(v) = d$ and $n = |V|$. Then $\operatorname{diam}(G) \geq \frac{\log n}{\log d}$.

Proof. Pick $v \in V$; $\forall k$, $|B_v(k)| \leq d\,|B_v(k-1)|$ because each $w \in B_v(k-1)$ has at most d neighbors $\Rightarrow$ $|B_v(k)| \leq d^k$. If $D := \operatorname{diam}(G) \Rightarrow B_v(D) = G$, so

$$n = |G| = |B_v(D)| \geq d^D.$$

$\square$

Some special graphs

A graph G is a forest if it contains no nontrivial cycle. It's a tree if it's a connected forest.

Ex.

forest; each comp. is a tree

Ex. K_n, $n \geq 3$, is not a forest

If G contains a loop or multiple edges it's not a forest.

Lemma

T finite tree with at least 2 vertices $\Rightarrow$ T contains at least two vertices of deg 1 (leaves).

Proof.

$|V(T)| \geq 2 \Rightarrow$ since T is connected, there is a walk between any two vertices, so in particular $|E(T)| \geq 1$. Look at walks γ with distinct vertices and let $\gamma = (v_0, e_1, \dots, v_n)$ be a longest such walk.

Rmk: e edge w/ endpts $v_1, v_2 \Rightarrow (v_1, e, v_2)$ is one such walk $\Rightarrow \ell(\gamma) \geq 1$.

We claim that $\deg(v_0) = \deg(v_n) = 1$. Indeed, if not, say for ex. $\deg(v_0) > 1$ and let $w \neq v_1$ be adjacent to v_0 and e the edge between them.

If $w \neq v_i \ \forall i \Rightarrow (w, e, v_0, e_1, \dots, v_n)$ is a walk with distinct vertices longer than γ ↯

$\Rightarrow w = v_i$ for some $i \neq 1 \Rightarrow (w, e, v_0, \dots, v_i)$ is a nontrivial cycle ↯

$\square$

Cor.

$G \neq \emptyset$ finite conn. graph s.t. $\deg(v) \geq 2 \ \forall v \Rightarrow G$ contains a cycle.

Ex. 7 Prove the corollary.

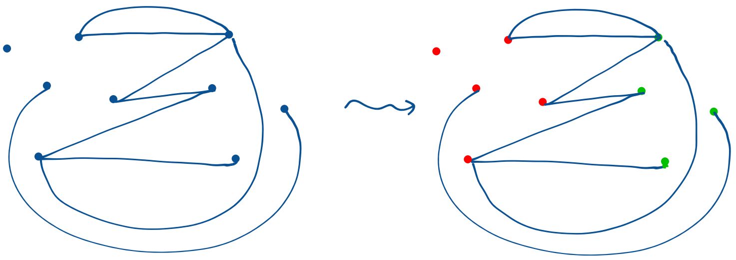
Ex. 8 G 2-regular, finite $\Rightarrow$ G is a cycle.

A graph G is bipartite if we can write $V(G)$ as $L \sqcup R$ so that $\forall e \in E(G)$ joins a vertex in L to a vertex in R. We call $\{L,R\}$ a bipartition of G.

This means that there are no edges with both endpoints in L, nor with both endpts in R.

We can think of bipartite graphs as graphs with vertices colored red or green and edges only joining vertices of different colors.

Ex.

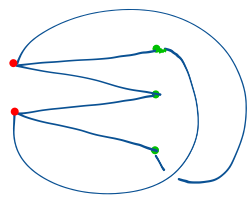


Ex. K_n: $n=2 \Rightarrow$ bipartite

$n \geq 3 \Rightarrow$ not bipartite: if we split $V = V_1 \sqcup V_2$, $V_i \neq \emptyset \Rightarrow$ at least one V_i has size $\geq 2 \Rightarrow \exists$ edge between the two vertices $\lightning$

Ex. complete bipartite graphs $K_{n,m}$, $n,m \geq 1$: $V(K_{n,m}) = L \sqcup R$, $|L| = n$, $|R| = m$ and there is 1! edge between any two vertices $v \in L$ and $w \in R$

$n=2$, $m=3$



Rmk: if a bipartition exists, it isn't always unique. For instances, vertices of degree zero (isolated vertices) can belong to any set of a bipartition.

Prop.
G graph. G bipartite $\Leftrightarrow$ G contains no closed walk of odd length.

Proof:

$\boxed{\Rightarrow}$ Let $\{L,R\}$ be a bipartition and $\gamma = (v_0, e_1, \dots, v_n)$ a closed walk.
Without loss of generality, assume $v_0 \in L$. As v_0 and v_1 are adjacent, $v_1 \in R$.
But then $v_2 \in L$, $v_3 \in R$ ... i.e. $v_i \in L$ if i even, $v_i \in R$ if i odd.
$v_n = v_0 \in L \Rightarrow n$ even, and n is the length of γ.

$\boxed{\Leftarrow}$ We use the hypothesis to construct a bipartition. Assume G is connected.
Pick $v \in V$. Define $L := \{w \in V \mid \exists$ walk from v to w of even length$\}$
$R := \{w \in V \mid \exists$ walk from v to w of odd length$\}$.

Claim 1: $L \sqcup R = V$.

Indeed, since G is connected, $\forall w\ \exists$ walk from v to $w \Rightarrow w \in L \cup R \Rightarrow L \cup R = V$.
Moreover, if by contradiction $L \cap R \neq \emptyset$, let $w \in L \cap R$. Then $\exists\ \gamma_1, \gamma_2$ walks from v to w, $\ell(\gamma_1)$ even and $\ell(\gamma_2)$ odd

$\Rightarrow \delta := \overline{\gamma_2} * \gamma_1$ closed walk and $\ell(\delta) = \ell(\gamma_1) + \ell(\gamma_2)$ odd $\lightning$

<u>Claim 2</u>: if e is an edge, it joins a vertex in L to a vertex in R.

Indeed, let $b(e) = \{v_1, v_2\}$. Wlog, $v_1 \in L \Rightarrow \exists\, \gamma$ walk from v to v_1 of even length.
$\delta := (v_1, e, v_2) \Rightarrow \delta * \gamma$ is a walk from v to v_2 of length $\ell(\gamma)+1 \Rightarrow$ odd $\Rightarrow v_2 \in R$.

By the two claims, $\{L, R\}$ is a bipartition.

<u>Ex. 8</u> (1) Show that G is bipartite iff each conn. comp. is bipartite.
(2) Use (1) to conclude the proof (for G not nec. connected).

$\square$