Exam

Basics in Mathematics: Geometry

Duration: 45 minutes. All answers need to be justified.
You can refer to results proven in class.

First name: Last name: Student number:

Exercise 1. Let G be a finite, d-regular graph. Prove that

$$|V(G)| = \frac{2}{d}|E(G)|$$

and deduce that if d is odd, $|V(G)|$ must be even.

Solution. By the handshaking lemma,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

As $\deg(v) = d$ for every v, we deduce that

$$d|V(G)| = 2|E(G)|$$

i.e.

$$|V(G)| = \frac{2}{d}|E(G)|.$$

In particular, 2 divides $d|V(G)| = 2|E(G)|$. If d is odd, 2 needs to divide $|V(G)|$, i.e. $|V(G)|$ is even.

Exercise 2. Let G be a finite, non-empty graph with n connected components. Show that $|E(G)| \geq |V(G)| - n$, with equality if and only if G is a forest.

Solution. We have seen in class that the result is true for connected graphs. If G has n connected components $C_1, \ldots, C_n$, then for every i

$$|E(C_i)| \geq |V(C_i)| - 1$$

with equality if and only if C_i is a tree.

Therefore

$$|E(G)| = \sum_i |E(C_i)| \geq \sum_i (|V(C_i)| - 1) = |V(G)| - n.$$

Moreover, we have equality if and only if we have an equality for every i, i.e. if and only if each component is a tree, i.e. if and only if G is a forest.

Exercise 3. Let G be a finite graph without loops with adjacency matrix A. Let $k \geq 1$ be an integer. Prove that $\mathrm{Tr}(A^k) = 0$ if and only if there are no closed walks of length k.

Solution. By definition of the trace,

$$\mathrm{Tr}(A^k) = \sum_{v \in V(G)} (A^k)_{vv}.$$

As seen in class, $(A^k)_{vv}$ is the number of walks from v to v of length k, i.e. the number of closed walks of length k starting from v. In particular, $(A^k)_{vv} \geq 0$ for every v, so

$$0 = \mathrm{Tr}(A^k) = \sum_{v \in V(G)} (A^k)_{vv}$$

if and only if $(A^k)_{vv} = 0$ for every v, i.e. if and only if there are no closed walks of length k starting at v for every v, i.e. if and only if there are no closed walks of length k.

Exercise 4. Let G be a graph whose Laplacian is the matrix

$$L = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

1. Is G connected?

2. Suppose furthermore that G is 4-regular. What is its adjacency matrix? Draw G.

Solution.

1. We can either compute the eigenvalues of L by looking at the characteristic polynomial and show that there are three distinct (real) eigenvalues, or check that the eigenvalue zero has eigenspace of dimension one. We deduce that 0 is an eigenvalue of multiplicity one, which implies — by a result seen in class — that G is connected.

2. The adjacency matrix A is $D - L$, where $D = \mathrm{diag}(4, 4, 4)$, so

$$L = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

So the graph G has three vertices v_1, v_2 and v_3, there is one loop based at v_1 and one loop based at v_2 and there are two edges between v_1 and v_2 and two edges between v_2 and v_3: