Exam

Basics in Mathematics: Geometry

Duration: 45 minutes

Exercise 1. Let G be a connected finite graph containing at least two vertices.

1. Prove that if G contains no vertex of degree one, it contains a cycle.
2. Use the previous result to show that if G is 2-regular, it is a cycle.

Solution.

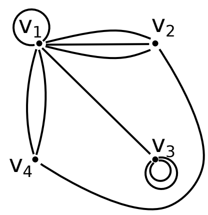
1. If by contradiction G doesn't contain a cycle, since it is connected, it is a tree. As seen in class, a tree with at least two vertices contains at least two leaves, i.e. at least two vertices of degree one, a contradiction.
2. By part 1, G contains a cycle C. Suppose $G \neq C$. If $V(G) = V(C)$, G must contain an edge $e \notin E(C)$ with an endpoint v on the cycle. But then the degree of v is at least three, a contradiction. So $V(G) \neq V(C)$, i.e. there is $v \in V(G) \setminus V(C)$. As G is connected, there is a path $\gamma = (v = v_0, e_1, \ldots, e_n, v_n = w)$ from v to some vertex $w \in V(C)$. Let i be the smallest integer so that $v_{i-1} \notin V(C)$ and $v_i \in V(C)$. Then the degree of v_i is at least three, a contradiction.

Exercise 2. Can the following matrices be adjacency matrices of a graph? If yes, construct a graph whose adjacency matrix is the given one. If not, explain why.

$$A = \begin{pmatrix} 2 & 5 & 6 & 0 \\ 5 & 0 & 0 & 1 \\ 6 & 0 & 1 & 4 \\ 0 & 1 & 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 2 & 0 \\ 1 & 3 & 0 & 4 \end{pmatrix}$$

Solution. A cannot be the adjacency matrix of a graph since it has an odd number on the diagonal (and by definition the (i,i) entry of the adjacency matrix is twice the number of loops based at v_i, which is even).

B is the adjacency matrix of the graph:



C cannot be the adjacency matrix of a graph since it is not symmetric (and by definition, the (i,j) entry, for $i \neq j$, is the number of edges from v_i to v_j, which is the same as the number of edges from v_j to v_i, and this is the (j,i) entry).

Exercise 3. Let $n \in \mathbb{N}$ and $p \in]0,1[$. Recall the model of random graph seen in class:

$$\mathcal{G}(n) := \{G \mid G \text{ simple graph with } V(G) = \{1, \dots, n\}\}$$

and

$$\mathbb{P}_{n,p}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|}.$$

1. Compute $\mathbb{P}_{n,p}(\text{the vertex } j \text{ has degree zero})$.
2. Prove that $\mathbb{P}_{n,p}(G \text{ has a vertex of degree zero}) \leq n(1-p)^{n-1}$.

Solution.

1. The vertex j has degree zero if and only if G doesn't contain any edge of the form $\{j,k\}$, for $k \neq j$. So

$$\mathbb{P}_{n,p}(\text{the vertex } j \text{ has degree zero}) = \mathbb{P}_{n,p}(E(G) \subset [V]^2 \backslash \{\{j,k\} \mid k \neq j\}) = (1-p)^{n-1}$$

   where the last equality follows from a result proven in class.

2. $\{G \text{ has a vertex of degree } 0\} = \bigcup_{j=i}^{n} \{G \mid j \text{ has degree zero}\}$, so

$$\mathbb{P}_{n,p}(G \text{ has a vertex of degree } 0) \leq \sum_{j=1}^{n} \mathbb{P}_{n,p}(j \text{ has degree zero}) = n(1-p)^{n-1}.$$