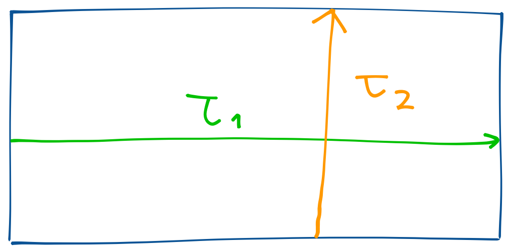
Translation surfaces

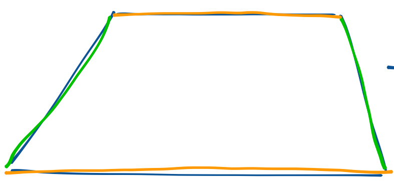
① Gluing polygons

Basic ex: the torus



Glue opposite sides of a rectangle using translations

Not:

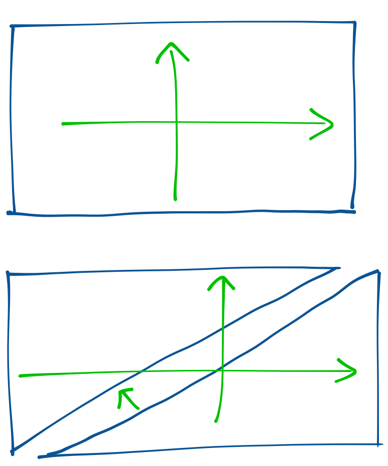


→ get the same topological surface, but not w/ the same structure

EX: there is a well-defined notion of North in the first case, but not in the second

"transporting" ↑ around following the id does not keep the direction

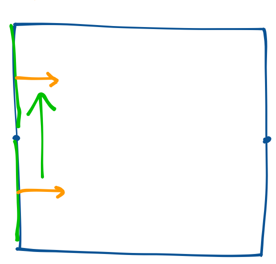
We want:



→ same surface

A translation surface is an equivalence class of finitely many polygons in $\mathbb{R}^2$ w/ edges identified by translation s.t. the inward normals of glued edges point in opposite directions. Each edge must be identified w/ exactly another edge.

↳ not allowed:



We require the quotient to be connected.

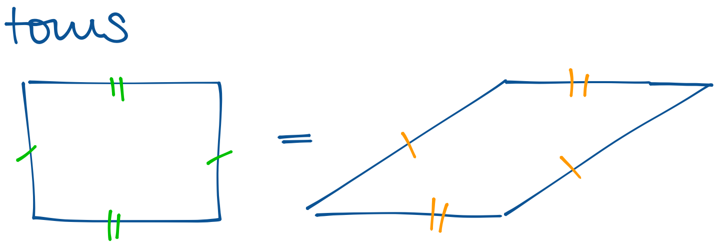
Rmk: a polygon is a closed region bounded by a piecewise linear Jordan curve. In particular, internal angles can be any number in $(0, 2\pi)$.

Rmk: we obtain a compact surface as (top.) quotient

Equivalence: two collections of polygons w/ sides identified are equivalent if one can be cut into pieces using straight lines and the pieces can be translated and reglued to form the other collection.

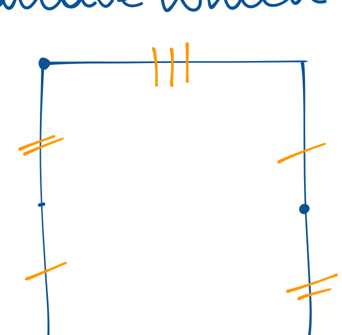
Each cut gives a new identification and only existing identif. can be used.

EX (1) the torus



Rmk: if ∃ translation sending one side to another, it is unique → we just indicate which sides are identified

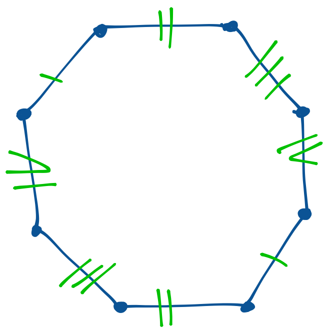
EX (2)



→ which top. surface is it?

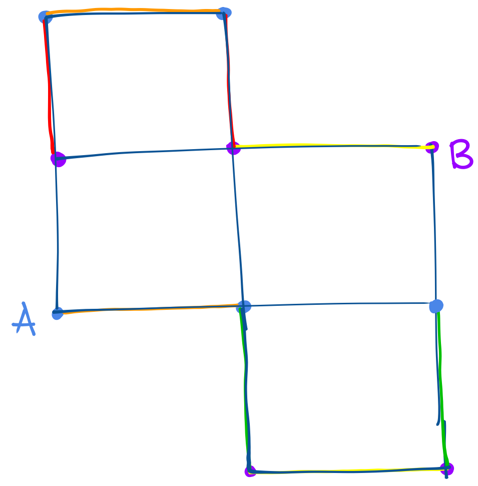
Compute Euler characteristic: 2 vertices, 3 edges and one face → $2-3+1 = 0 = 2-2g \rightsquigarrow$ it is a torus

EX (3): not a torus



$\chi(S) = 1 - 4 + 1 = -2 \rightsquigarrow \underline{g = 2}$

EX (4)



$\chi(S) = 2 - 5 + 1 = -2 \Rightarrow g = 2$

Exercise: show $\exists$ translation surfaces of any genus $g \geq 1$.

? $g = 0$?

$\to$ let's examine the geometry.

$p \in X$ singular if the total angle around it is $\neq 2\pi$; regular otherwise.

Rmk: only vertices of polygons can be singular, though not all are

In the ex: (1) no sing pts

(2) one, angle 6π

(3) two, angles 4π

Connection btw geom. & top.:

Prop. (Gauss-Bonnet theorem for translation surfaces)

X w/ singular points of angles $\alpha_1, \dots, \alpha_n$. Then

$$\chi(X) = \frac{1}{2\pi} \sum_{i=1}^{n} (2\pi - \alpha_i).$$

Proof. Start w/ polygons w/ edge gluings and triangulate them $\longrightarrow$ get a triangulation of X w/ v vertices, e edges and f faces.

Note that $2e = 3f$: each face has 3 edges and each edge is part of two faces.

Let's compute the sum of all internal angles of the triangles:

* it is πf (they are Euclidean triangles)

* it is $\displaystyle\sum_{p \text{ vertices}} \underbrace{\alpha_p}_{\text{total angle around } p} = \sum_{\text{sing pts}} \alpha_p + \sum_{\text{nonsing pts}} \alpha_p = \sum_{i=1}^{n} \alpha_i + (v-n) 2\pi$

$$\Rightarrow f = 2(v-n) + \sum_{i=1}^{n} \frac{\alpha_i}{\pi}$$

Then:

$$\chi(X) = v - e + f = v - \frac{1}{2} f = v - (v-n) - \sum_{i=1}^{n} \frac{\alpha_i}{2\pi} = -\sum_{i=1}^{n} \frac{\alpha_i}{2\pi} + n.$$

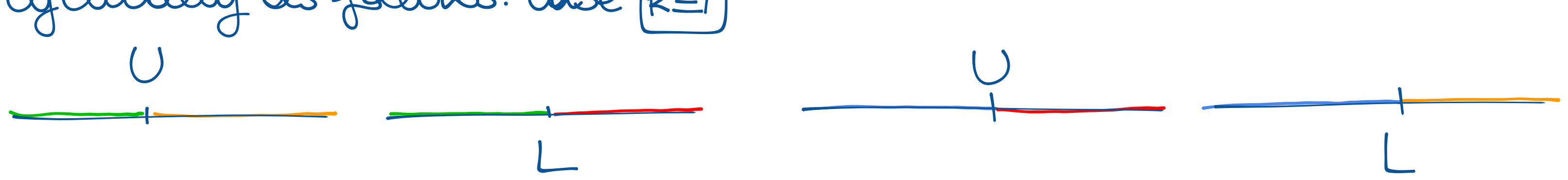
$\square$

Next we want to show that actually angles at singular pts can only be mult. of 2π.

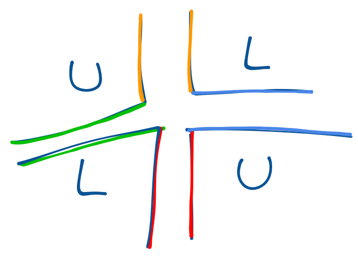
Note: if we know this, we get $\chi(X) \leq 0 \Rightarrow$ a translation surface cannot be a sphere.

We will actually prove more.

C_{k+1} is obtained from $(k+1)$ copies of $U = \{\operatorname{Im} z \geq 0\}$ and $(k+1)$ copies of $L = \{\operatorname{Im} z \leq 0\}$ glued cyclically as follows: case $\boxed{k=1}$



Top:



<u>Note</u>: $\forall q \in C_{k+1}, q \neq 0$, $\exists$ nbhood $\mathcal{U}$ of q which is either contained in a single copy of U or L or in the union of a copy of U and a copy of L glued along a half real line. So we can see $\mathcal{U}$ as a subset of either $\{\operatorname{Im} z < 0\}$, $\{\operatorname{Im} z > 0\}$, $\{\operatorname{Re} z < 0\}$ or $\{\operatorname{Re} z > 0\}$ and we get an isometric embedding of $\mathcal{U}$ into $\mathbb{C}$. We denote this set of charts of $C_{k+1} \setminus \{0\}$ by B_{k+1}.

<u>Def</u>. S top. surface. A <u>translation structure</u> on S is the datum of
- $\Sigma = (p_1, \dots, p_n)$ a finite set of pts of S (the <u>singular points</u>)
- $k = (k_1, \dots, k_n)$ a collection of positive integers (the <u>multiplicities</u>)
- an atlas $\mathcal{A}$ of chart to $\mathbb{C}$ on $S \setminus \Sigma$
- $\forall J$, an open nbhood U_J of p_J and a homeomorphism $\varphi_J : U_J \longrightarrow$ open nbhood of $0 \in C_{k_J+1}$

such that all transition functions between charts in

$$\mathcal{A} \cup \bigcup_{J=1}^{n} (\varphi_J)_* B_{k_J+1}$$

are translations. Here $(\varphi_J)_* B_{k_J+1} = \left\{ \left(\varphi_J^{-1}(\varphi_J(U_J) \cap U), \psi \circ \varphi_J \right) \,\middle|\, \begin{matrix} (U, \psi) \in B_{k_J+1}, \\ \varphi_J(U_J) \cap U \neq \emptyset \end{matrix} \right\}$

<u>Prop.</u>

Let X be a translation surface. Then this gives a translation structure on X where Σ is the set of singular points and if $p \in \Sigma$, its multiplicity is $\frac{\alpha_p}{2\pi} - 1$. In particular, for every singular point p, α_p is an integer multiple of 2π.

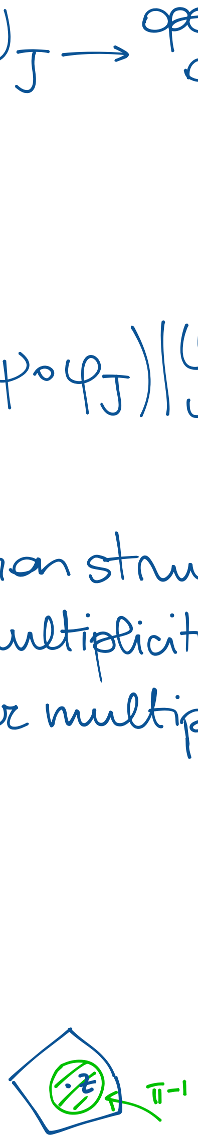
<u>Proof</u>. Let's first construct the chart in $\mathcal{A}$.

$p \in X$, $\pi : \bigsqcup P_i \longrightarrow X$ projections, p non singular

(some) polygons defining X

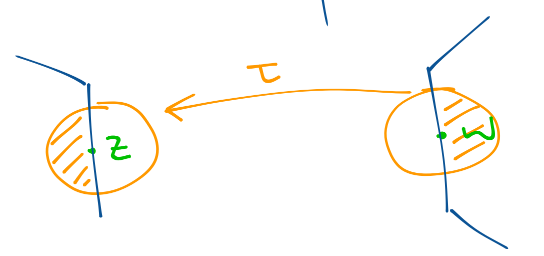
3 cases:

(1) $p = \pi(z)$, $z \in \operatorname{int}(P_J)$ for some $J \Rightarrow \exists \varepsilon > 0 : B_\varepsilon(z) \subseteq \operatorname{int}(P_J)$.



Set $(U_p, \varphi_p) = \left(\pi(B_\varepsilon(z)), \left(\pi|_{\pi(B_\varepsilon(z))}\right)^{-1} \right)$ chart around p.

(2) $p = \pi(z)$, $z \in$ side of P_J, not a vertex.



Then: P_k

$\exists (!)\, \underset{\in}{w} \neq z : \pi(w) = p$. Denote by τ the translation mapping the edge containing w to the edge containing z. Let $\varepsilon > 0$ s.t. $B_\varepsilon(z)$ and $B_\varepsilon(w)$ do not contain vertices of polygons.

Set $U_p := U_z \cup U_w$ where $U_z := \pi(B_\varepsilon(z) \cap P_J)$, $U_w := \pi(B_\varepsilon(z) \cap P_k)$.

$\varphi_p : U_p \longrightarrow \mathbb{C}$ given by

$\varphi_p(q) = \pi^{-1}(q) \cap P_J$ if $q \in U_z$

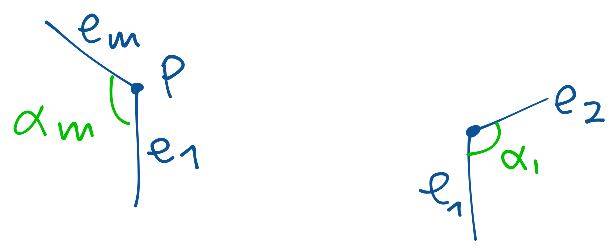
$\varphi_p(q) = \tau(\pi^{-1}(q) \cap P_k)$ if $q \in U_w$

<u>Rmk</u>: if $q \in U_z \cap U_w \Rightarrow \pi^{-1}(q)$ is given by two pts, one in $\partial P_J \cap B_\varepsilon(z)$ and one in $B_\varepsilon(w) \cap \partial P$, and τ sends the second to the first.

$\rightsquigarrow (U_p, \varphi_p)$ local chart at p

(3) $p = \pi(z)$, z vertex: similar to (2), just more complicated (there are more preimages to be considered, but still finitely many $\Rightarrow$ get ε and repeat the procedure) $\rightsquigarrow (U_p, \varphi_p)$

Next we check: all angles at singular points are multiples of 2π. So let $p \in \Sigma$ and call the images of the edges emanating from it $e_1, \dots, e_m$ (following the circular counterclockwise order). Up to rotating the plane and shifting the numbering of the edges by 1 we can assume e_1 is vertical and goes down from p. In pictures:



Denote by α_i the angle between e_i and e_{i+1} (where $e_{m+1} = e_1$)

Since we have a translation surface, the other copy of e_1 is parallel to the first copy. Rotate clockwise by α_1 to bring e_2 vertical and south of p:

Rotating again by α_2 we get that e_3 is now vertical and south of p. Note that we rotated by $\alpha_1 + \alpha_2$ in total. Repeat until

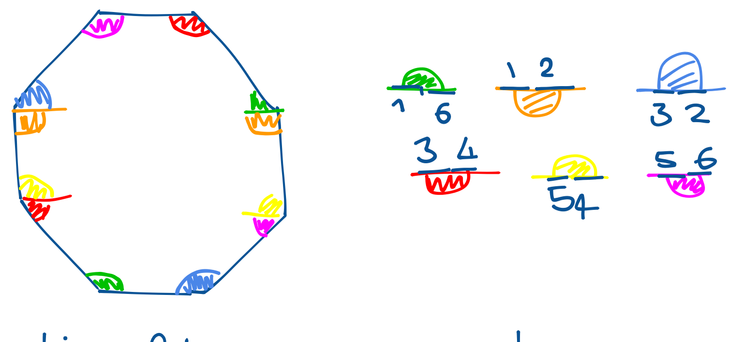
So far, we rotated by $\alpha_1 + \dots + \alpha_{n-1}$. By rotating by α_n we go back to e_1 in the same direction as at the start $\Rightarrow$ the rotation must be of a multiple of 2π

$$\underset{\alpha_p}{\underset{\|}{\alpha_1 + \dots + \alpha_n}} = k 2\pi$$

To get the homeomorphism from the neighborhood of p: choose ε so that there is no other vertex at distance $< \varepsilon$ from any preimage of p in $\sqcup P_i$. Claim: the ε-nbhood of p in X is isometric to the ε-nbhood of 0 in C_{k+1}, where

$$k = \frac{\alpha_p}{2\pi} - 1.$$

EX.



Transition fcts: any chart we constructed sends a nbhood of a regular point of X to a neighborhood of one of its preimages or to a translate of such a neighborhood. Since two preimages of a pt differ by a translation, we get that the transition maps are translations. $\square$

Exercise: write down the homeo around the singular points.

Actually, the conditions of the proposition characterize translation surfaces. We'll go through another definition to show this.

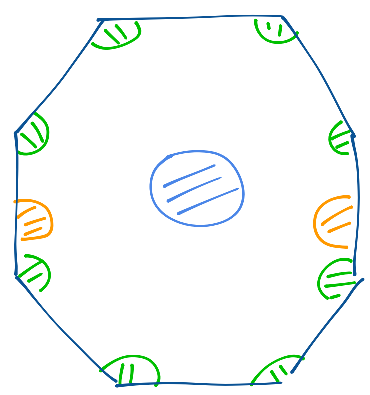
② Riemann surfaces and Abelian differentials

A <u>translation surface</u> is a pair (X,ω), where X is a cpt Riemann surface and ω is a non-zero Abelian differential.

<u>EX. (1)</u> X torus, $\omega = \lambda \omega_0$ ← Abelian diff. $=1$ in loc charts

<u>Ex.(2)</u> $\nexists$ non-zero Abelian differential on the sphere → we find again that a translation surface must have genus at least 1

<u>Ex (3)</u> On a genus 2 surface; $\Sigma = \{p_0\}$



In the local charts that we described:
(U,φ) around $p \neq p_0$ ⇝ set $\omega = 1$

Around p_0: map $U_{p_0} \xrightarrow{\varphi_p}$ union of 6 half planes $= C_3$
(3 U and 3 L)

want this to be given by $z \mapsto \frac{z^3}{3}$ $\quad\mathbb{C}$

<u>Idea</u>: this gives a branched cover of $\mathbb{C}$ of degree 3, but C_3 is given by multiple copies ⇝ by mapping z to the "correct copy" of U or L we get a homeomorphism φ_3
⇒ we invert it

The chart at p_0 is then $(U_{p_0}, \varphi_3^{-1} \circ \varphi_{p_0})$

Set ω: 1 in the charts around $p \neq p_0$
z^2 in the chart around p_0

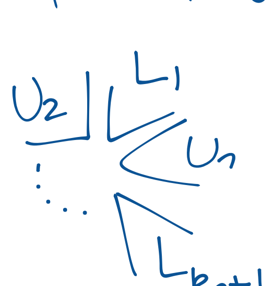
(U,φ) around $p \neq p_0$, $(U_{p_0}, \varphi_3^{-1} \circ \varphi_{p_0})$ ⇝ $w(z) = z^3/3 \Rightarrow z^2 = \frac{dw}{dz} \cdot 1$
↓ loc coord: w $\qquad$ ↓ loc coord: z

With the same idea we will show that a translation structure gives us a cpx structure and an Abelian differential:

<u>Prop.</u>
A translation structure on a surface S induces a Riemann surface structure X on S and a nonzero Abelian differential ω on X such that the zeroes of ω are the singular points of the translation structures and the order of zero of ω is the multiplicity of the singular point.

<u>Proof</u>: The translation structure gives us in particular an atlas of charts of $S \setminus \Sigma$ to $\mathbb{C}$ with transition functions that are translations and hence holomorphic. Around a singular point p of multiplicity k_p:

$$\varphi: U_p \longrightarrow V_p \subseteq C_{k_p+1} =$$



$\forall i,\ L_i, U_i \subseteq \mathbb{C}$

Write $\mathbb{C}$ as the union $\bigcup_{J=1}^{2(k_p+1)} S_J$, where

$$S_J = \left\{ z = re^{i\theta} \,\middle|\, \theta \in \left[\frac{\pi}{k_p+1} \cdot J,\ \frac{\pi}{k_p+1}(J+1)\right] \right\}.$$

In words: these are the sectors of angle $\frac{\pi}{k_p+1}$.

Define:

- if J even, $S_J \longrightarrow L_{J/2} \simeq L \subseteq \mathbb{C}$
$$z \longmapsto \frac{z^{k_p+1}}{k_p+1}$$
- if J odd, $S_J \longrightarrow U_{\frac{J+1}{2}} \simeq U \subseteq \mathbb{C}$
$$z \longmapsto \frac{z^{k_p+1}}{k_p+1}$$

We glue these to get a map $\psi_{k_p+1} : \mathbb{C} \longrightarrow C_{k_p+1}$. This is a homeomorphism, so we can define a chart $\psi_{k_p+1}^{-1} \circ \varphi_p : U_p \xrightarrow{\theta_p} \mathbb{C}$.

Claim: $\mathcal{A} \cup \{(U_p, \theta_p) \mid p \in \Sigma\}$ gives a cpx structure on S.

Indeed: the transition fct between any two charts in $\mathcal{A}$ is a translation, so it is holomorphic. We just need to check that the change of coordinate btw a chart around a singular point and a chart in $\mathcal{A}$ is a biholomorphism. So fix p and a chart (V, φ) w/ $V \cap U_p \neq \phi$. Up to restricting V, we can assume that $\varphi_p(V)$ is contained in the open set corresponding to a chart in $(\varphi_p)_* B_{k_p+1}$, say (W, θ). We have
$$\varphi \circ (\psi_{k_p+1}^{-1} \circ \varphi_p)^{-1} = \underbrace{\varphi \circ \theta^{-1}}_{\substack{\text{translation} \\ \Rightarrow \text{biholomorphic}}} \circ \underbrace{\theta \circ \varphi_p^{-1} \circ \psi_{k_p+1}}_{\substack{z \mapsto \frac{z^{k_p+1}}{k_p+1} \\ \text{biholomorphic (since away from 0)}}}$$

To define ω: associate the fct 1 to each chart in $\mathcal{A}$ and the function z^{k_p} to the chart (U_p, θ_p).

Check that this defines an Abelian differential:

- on overlapping charts in $\mathcal{A}$, the derivative of the transition fct is 1, so the compatibility condition is satisfied
- consider (U_p, θ_p) and (V, ψ) w/ $U_p \cap V \neq \phi$. The transition fct is

$$\underset{\text{coord } z}{\downarrow} \qquad \underset{\text{coord } w}{\downarrow}$$
$$w(z) = \frac{z^{k_p+1}}{k_p+1} + c \implies \frac{dw}{dz} = z^{k_p} \text{ and } z^{k_p} = \frac{dw}{dz} \cdot 1 \text{, as required.}$$

$\square$

Given (X, ω), as in the proof above we define charts into $\mathbb{C}$ given by: $\forall p \in X$ zero of order $k \geq 0$ (where $k=0$ means that p is not a zero) chose a small contractible nbhood U_p of p not containing any $q \neq p$ zero of order ≥ 1; $\varphi_p : U_p \longrightarrow \mathbb{C}$
$$q \longmapsto \left(\int_p^q \omega\right)^{1/k+1}.$$

Note: in this chart, ω is of the form $(k+1) z^k$

If $k=0$: use φ_p to define a metric on U_p by pulling back the Euclidean metric on $\mathbb{C}$.

If $k \geq 1$: consider $U_p \setminus \{p\} \xrightarrow{\varphi_p} \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$
$$z \longmapsto z^{k+1}$$
and pull back the Euclidean metric on $U_p \setminus \{p\}$.

Lemma

The metrics define a global metric on $X \setminus \Sigma$, $\Sigma = \{\text{zeroes of } \omega\}$.

Proof: we need to show that they give the same metric on the overlap of two charts. So let $p \in X \setminus \Sigma$ and $q \in X$ and let $x, y \in U_p \cap U_q$. Then the distance between x and y as measured on U_p is $d_{\mathbb{C}}\left(\int_p^x \omega, \int_p^y \omega\right)$. In U_q it is

Euclidean dist. on $\mathbb{C}$

$$d_{\mathbb{C}}\left(\left(\left(\int_q^x \omega\right)^{\frac{1}{k_q+1}}\right)^{k_q+1}, \left(\left(\int_q^y \omega\right)^{\frac{1}{k_q+1}}\right)^{k_q+1}\right) = d_{\mathbb{C}}\left(\int_q^x \omega, \int_q^y \omega\right) = d_{\mathbb{C}}\left(\int_p^x \omega, \int_p^y \omega\right).$$

$\square$

This gives us then a flat metric on $X \setminus \Sigma$, where a flat metric is a Riemannian metric of constant curvature 0. Equivalently, $X \setminus \Sigma$ with this metric is locally isometric to $\mathbb{C}$ with the Euclidean metric.

Note that we cannot hope to have a flat metric on the whole of X, unless X is a torus:

Thm (Gauss-Bonnet thm for Riemannian surfaces)
Let F be a cpt Riemannian surface of Gauss curvature K. Then

$$\int_F K \, dA = 2\pi \chi(F).$$

So:
- $K \equiv 0 \Rightarrow \chi(F) = 0$, i.e. F is a torus
- $K \equiv 1 \Rightarrow F$ sphere
- $K \equiv -1 \Rightarrow F$ has genus at least 2.

Aside:
- $\exists$ metric with $K \equiv 1$ on S^2; it is unique up to isometry
- $\forall g \geq 2$, $\exists$ metric with $K \equiv -1$ on a surface of genus g. Actually, there are many (even up to isometry)

hyperbolic metric

Let's try to understand better what happens around $p \in \Sigma$.
We want to understand what lengths of curves are. Then for any x, y the distance can be described as

$$d(x,y) = \inf\{\underbrace{\boxed{\ell(\gamma)}}_{\text{length of } \gamma} \mid \gamma \text{ curve from } x \text{ to } y\}.$$

Recall that if $f: U \to \mathbb{C}$ is a local diffeo and U is endowed with the pull-back metric, then for $\gamma: I \to U$ a curve

$$\underbrace{\ell_U}_{\substack{\text{length measured}\\ \text{wrt pull-back}\\ \text{metric}}}(\gamma) = \underbrace{\ell_{\mathbb{C}}}_{\text{length measured in } \mathbb{C}}(f \circ \gamma)$$

In our case we have: $\overbrace{B_\varepsilon(0) \setminus \{0\}}^{U} \xrightarrow{f} \mathbb{C} \setminus \{0\}$
$z \longmapsto z^{k+1}$

Ex 1: $\gamma(t) = t$, $t \in (0, \varepsilon)$

Then $\ell_U(\gamma) = \ell_{\mathbb{C}}(f \circ \gamma) = \int_0^\varepsilon \|(\gamma(t)^{k+1})'\| \, dt = \int_0^\varepsilon \|(k+1)\gamma(t)^k \gamma'(t)\| \, dt = \int_0^\varepsilon (k+1) t^k \, dt = \varepsilon^{k+1}$

Finite length $\Rightarrow$ the metric is not complete.

Ex 2 $\gamma(t) = \frac{\varepsilon}{2}(\cos t, \sin t)$, $t \in [0, 2\pi]$

$$\ell_U(\gamma) = \int_0^{2\pi} \left(\frac{\varepsilon}{2}\right)^{(k+1)} \underbrace{\|\gamma(t)^k \gamma'(t)\|}_{\substack{\| \\ 1}} (k+1) \, dt = 2\pi(k+1) \underbrace{\left(\varepsilon/2\right)^{k+1}}_{\substack{\text{radius: distance from } 0\\ \text{(if we complete the metric)}}}$$

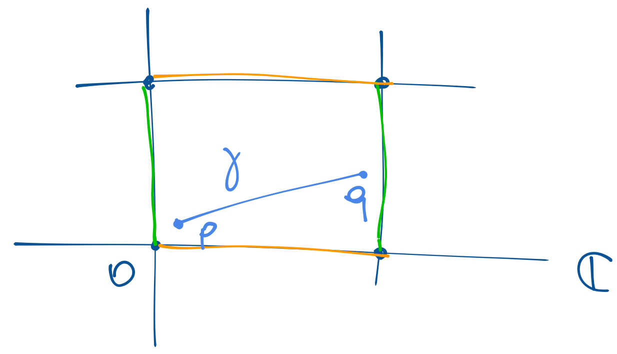
More than in Euclidean space $\rightsquigarrow$ this suggests that we have "angle $2\pi(k+1)$" at the singularity.

Indeed, $B_\varepsilon(0)\setminus\{0\}$ with this metric is isometric to a nbhood of the origin in C_{k+1} and the map ψ_{k+1} constructed in a proof before provides the isometry.

Let M be a metric space. A curve $\gamma: I\to M$ is a <u>geodesic</u> if it is locally length minimizing, i.e. $\forall t\in I\ \exists\varepsilon>0$ s.t. $\forall r<s\in[t-\varepsilon,t+\varepsilon]\cap I$

$$d_M(\gamma(r),\gamma(s))=\ell(\gamma|_{[r,s]}).$$

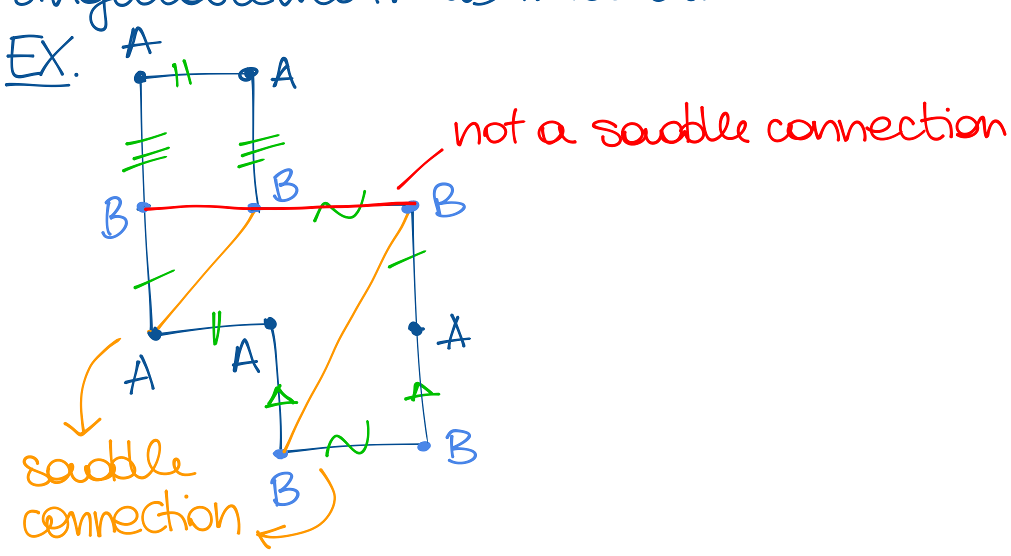
<u>Rmks</u>:
- parametrization is not important in this definition of geodesic
- it is important that we require <u>local</u> length minimization:



The light blue curve is a geodesic, but $d(p,q)\neq\ell(\gamma)$.

Let X be a Riemann surface with $\omega\not\equiv 0$ Abelian differential.
A <u>saddle connection</u> for (X,ω) is a geodesic for the associated singular flat metric joining two singularities (which can coincide) and without singularities in its interior.

<u>EX</u>.

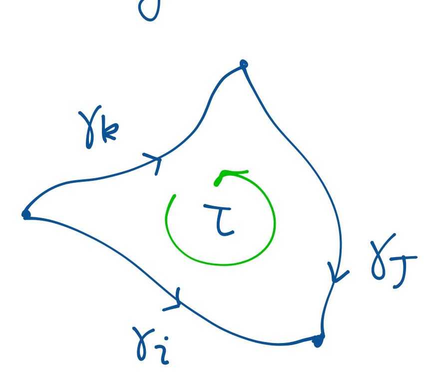


<u>Prop.</u>
For any (X,ω) of genus at least two, there exists a triangulation of X made by saddle connections whose set of vertices is the set of zeroes of X.

We use this to show:
<u>Prop.</u>
X Riemann surface, $\omega\not\equiv 0$ Abelian differential. Then $\exists$ an associated transl. surface Y (given as gluings of polygons) and a homeo $Y\simeq X$ s.t. the zeroes of ω correspond to the singularities of Y and the cone angles are given by the order of zeroes: if $q\in Y$ corresponds to $p\in X$, then

$$\alpha_q=2\pi(k_p+1).$$

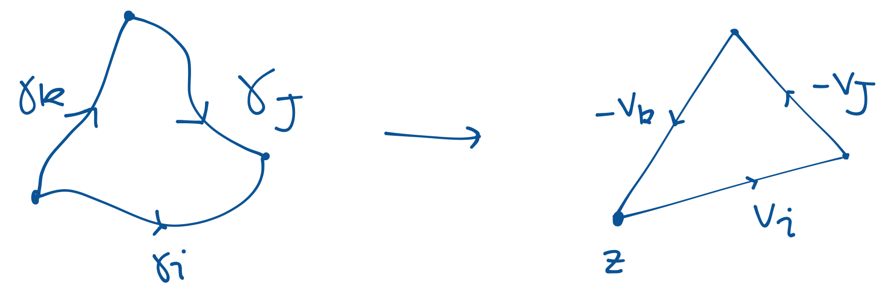
<u>Proof</u>. If X is a torus, there is nothing to prove. Suppose then that the genus of X is at least two. Then $\exists$ a triangulation made of saddle connections as in the previous proposition. Fix an orientation on X and let $\gamma_1,\dots,\gamma_N$ be parametrization of the edges of the triangulation.
For any triangle τ, $\partial\tau=\gamma_k^{\pm1}*\gamma_j^{\pm1}*\gamma_i^{\pm}$ (where the orientation of τ and $\partial\tau$ is induced by the orientation of X).



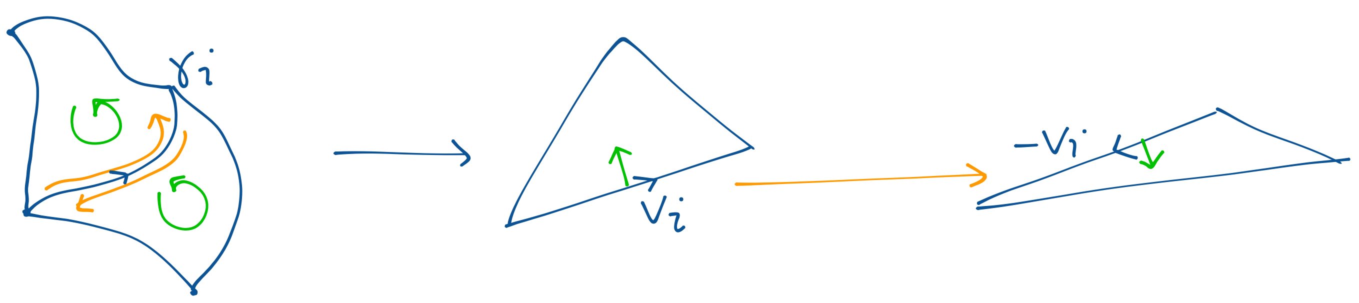
$$\partial\tau=\gamma_k^{-1}*\gamma_j^{-1}*\gamma_i$$

Set $v_\ell:=\int_{\gamma_J}\omega\in\mathbb{C}$.

Since τ is a topological disk, $\int_\tau \omega = 0$. Hence $\pm v_i \pm v_J \pm v_k = 0$, which means that the vectors give a triangle in $\mathbb{R}^2$:



To each triangle associate a Euclidean triangle in $\mathbb{R}^2$ as above, putting them far enough from each other so that they don't overlap. Note that $\forall i$, γ_i belongs to two adjacent triangles with opposite orientations, so we have two parallel sides of triangles in $\mathbb{R}^2$ that can be identified by a translation. The condition on the orientation shows that the inner normals are glued following the original translation surface definition.



So we get a translation surface Y (polygon definition). X and Y have the same triangulation, so identifying the triangles via homeomorphisms respecting the side identifications gives us a homeomorphism $Y \simeq X$ sending vertices to vertices. As only vertices of polygons give singularities of Y, we get that

$$\{\text{sing. of } Y\} \subseteq \{\text{zeroes of } \omega\}.$$

identifying
X and Y
via the homeo

One can show that the angles between the v_i's are the same as the angles between the saddle connection in the singular flat structure, which shows that $\forall p$ vertex $\alpha_p = 2\pi(k_p+1)$ and ends the proof.

□