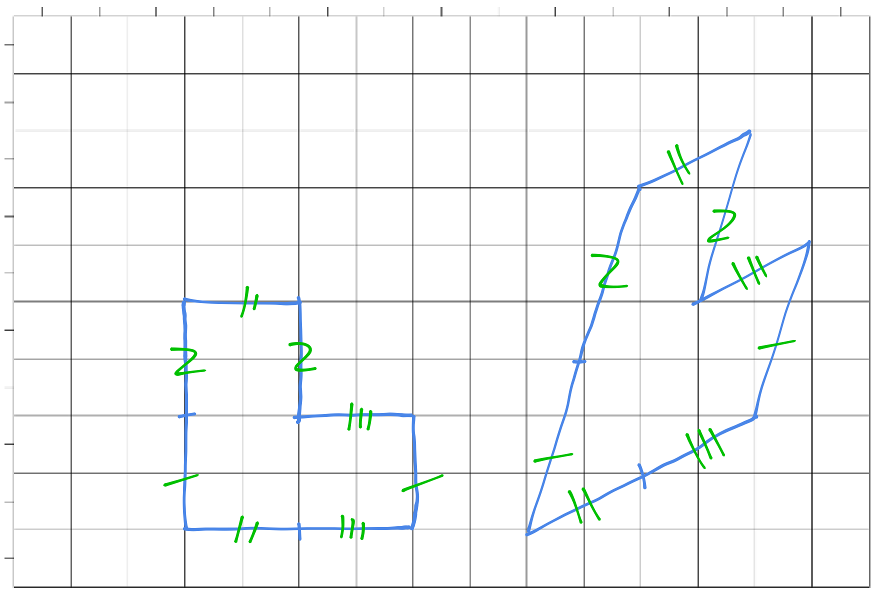
The $GL^+(2,\mathbb{R})$-action

There is a natural action of $GL^+(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}) \mid \det A > 0\}$ on $\mathcal{H}(k)$ which plays a very important role in the study of translation surfaces.
The most intuitive way to define it is looking at gluings of polygons.
If a translation surface X is given by the gluing of polygons $P_1, \dots, P_m \subseteq \mathbb{R}^2$ and $A \in GL^+(2,\mathbb{R})$, $A \cdot X$ is the surface given by the gluing of $A \cdot P_1, \dots, A \cdot P_m$ where the action on the polygons is induced by the standard action of $GL^+(2,\mathbb{R})$ on $\mathbb{R}^2$. This action sends parallel vector of the same lengths to parallel vectors of the same length, so $A \cdot X$ is indeed a translation surface.

EX



Since this is induced by the action on $\mathbb{R}^2$, we have an actual group action, i.e. $\forall A, B \in GL^+(2,\mathbb{R})$, $\forall X \in \mathcal{H}(k)$

(1) $Id \cdot X = X$

(2) $A \cdot (B \cdot X) = \underbrace{(AB)}_{\text{matrix mult.}} \cdot X$.

action action

The action can be described also for Riemann surfaces and Abelian differentials: given $[(X,\omega)] \in \mathcal{H}(k)$, $A \cdot [(X,\omega)]$ is given by the Riemann surface defined by the atlas $\{(U, \underbrace{A \circ \varphi}) \mid (U,\varphi) \text{ chart for } X\}$

$$U \xrightarrow{\varphi} \mathbb{C} \simeq \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \simeq \mathbb{C}$$

and Ab. differential: to the chart $(U, A\circ\varphi)$ we associate $f \circ A^{-1}$, where f is the function associated to ω in the chart (U,φ).

For singular flat structures: X translation structure $[(\Sigma, k, \mathcal{A}, \{(U_p, \varphi_p)\})]$ on $S \Rightarrow$ define $A \cdot \mathcal{A}$ on $S \setminus \Sigma$ and take the metric completion to recover the structure on the full surface.

Note that $\underbrace{GL^+(2,\mathbb{R})}_{\text{top. as subspace of } Mat(2\times 2, \mathbb{R}) \simeq \mathbb{R}^4}$ acts continuously on $\mathcal{H}(k)$: one can check it by looking at the action on period coords or on polygons

Given $X \in \mathcal{H}(k)$, the stabilizer of X is $Stab_{GL^+(2,\mathbb{R})}(X) := \{A \in GL^+(2,\mathbb{R}) \mid A \cdot X = X\}$.

Rmk: since we have a group action, this is a subgroup of $GL^+(2,\mathbb{R})$. Indeed:

- $Id \in Stab_{GL^+(2,\mathbb{R})}(X)$ by condition (1) of group actions
- $A \in Stab \Rightarrow X = A \cdot X \Rightarrow A^{-1} \cdot X = A^{-1} \cdot (A \cdot X) \overset{(2)}{=} (A^{-1}A) \cdot X \overset{(1)}{=} X$, i.e. $A^{-1} \in Stab$
- $A, B \in Stab \Rightarrow (AB) \cdot X \overset{(2)}{=} A \cdot (B \cdot X) \underset{B \in Stab}{=} A \cdot X \underset{A \in Stab}{=} X \Rightarrow AB \in Stab$.

Rmk: $\forall X$, $\mathrm{Stab}_{GL^+(2,\mathbb{R})}(X) \subseteq SL(2,\mathbb{R})$, because the area must be preserved, i.e. the determinant of A in the stabilizer must be $1 \Rightarrow \mathrm{Stab}_{GL^+} = \mathrm{Stab}_{SL}$.

In general, stabilizers and orbits have some common properties:

Prop.
The $SL(2,\mathbb{R})$-orbit of any $X \in \mathcal{H}(k)$ is unbounded. The stabilizer of any X is discrete but not cocompact.

Discrete = it has the discrete topology. Since it is a top. group (i.e. the group operations are continuous functions) it is equivalent to say that Id is isolated.

A subgroup H of a topological group G is cocpt if $\exists K \subseteq G$ cpt s.t. $KH = G$.
Equivalently, it is cocpt if the set of cosets $G/_H$ is cpt (endowed with the quotient top.).

EX: • $(\mathbb{Z},+)$ is a discrete group, while $(\mathbb{R},+)$ is not.
• $(\mathbb{Z},+)$ is cocpt in $(\mathbb{R},+)$, but not in $(\mathbb{C},+)$

$\downarrow$ quotient S^1 $\qquad$ $\downarrow$ quotient $S^1 \times \mathbb{R}$

Proof.
(1) Stabilizers are discrete:
Since the action is continuous, given a nbhood U of X on which the period coords are local coordinates, there is a nbhood V of Id s.t. if $A \in V$, $A \cdot X \in U$. Supp. $A \in U$ is not Id. The period coords of $A \cdot X$ are different than the period coordinates of X and since we are in a local chart this implies that $A \cdot X \neq X$, i.e. $A \notin \mathrm{Stab}(X)$. Hence $U \cap \mathrm{Stab}(X) = \{\mathrm{Id}\}$, i.e. the identity is an isolated pt and $\mathrm{Stab}(X)$ is discrete.

(2) Orbits are unbounded:
If X has genus ≥ 2, it has a saddle connection. If it has genus one, it has a simple closed geodesic. Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ s.t. $R_\theta \cdot X$ has a vertical saddle conn./ simple closed geodesic of length ℓ. Then for $G_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, the length of the corresponding saddle conn./simple closed geod is $e^{-t}\ell \longrightarrow 0$ as $t \to \infty$. So $\{G_t R_\theta(X)\}_{t>0}$ is unbounded and so the same holds for $SL(2,\mathbb{R}) \cdot X \supseteq \{G_t R_\theta(X)\}_{t>0}$.

(3) Stabilizers are not cocpt:
We have a continuous map

$$\begin{aligned} SL(2,\mathbb{R})/\mathrm{Stab}(X) &\longrightarrow \mathcal{H}(k) \\ A \cdot \mathrm{Stab}(X) &\longmapsto A \cdot X \end{aligned}$$

well-defined by def. of stabilizer

The image of this map is the $SL(2,\mathbb{R})$-orbit of X, which is not compact, as it is unbounded. Thus $SL(2,\mathbb{R})/\mathrm{Stab}(X)$ is not cpt. $\square$

Prop.
$\mathrm{Stab}_{GL^+(2,\mathbb{R})}(\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}, \omega_0) = SL(2,\mathbb{Z})$.

std (1 in std coord charts)

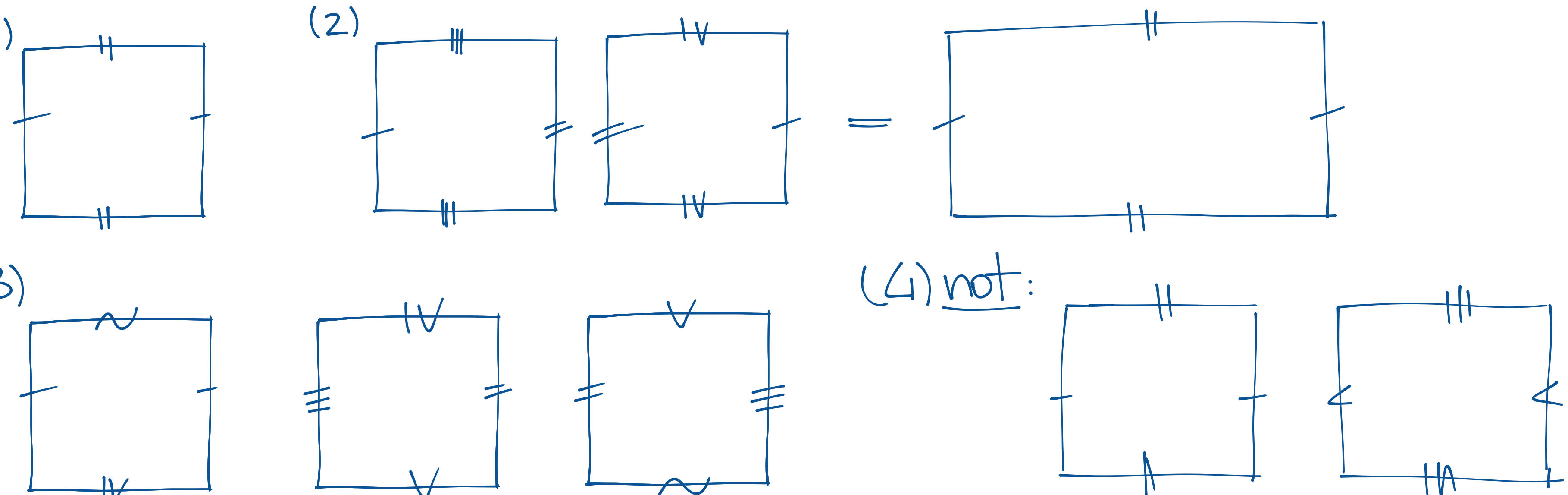
Proof. $A\cdot(\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}) = \mathbb{C}/A\cdot(\mathbb{Z}\oplus i\mathbb{Z}) \longrightarrow$ same surface iff $\mathbb{Z}\oplus i\mathbb{Z} = A\cdot(\mathbb{Z}\oplus i\mathbb{Z})$
i.e. iff $A\in SL(2,\mathbb{Z})$.

But the differential is 1 in std coords on both sides, i.e. if $A\in SL(2,\mathbb{Z})$
$$A\cdot(\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}, \omega_0) = (\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}, \omega_0)$$
So $A\in \mathrm{Stab}_{GL^+(2,\mathbb{R})}(\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}, \omega_0)$. $\square$

There is a larger class of translation surfaces for which we have a good understanding of their stabilizers.

A square-tiled surface is a translation surface obtained by gluing finitely many unit squares.

EX: (1) (2) (3) (4) not:



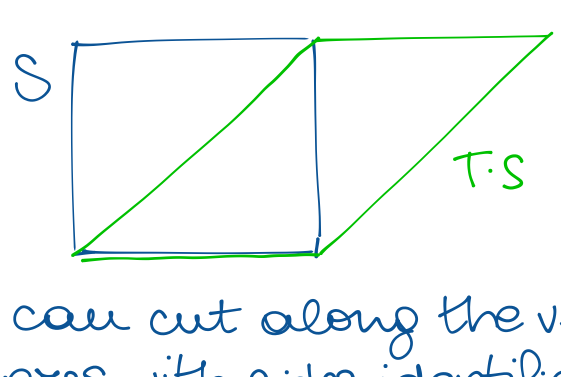
(not connected!)

Lemma
For any square-tiled surface X, $\mathrm{Stab}_{SL(2,\mathbb{R})}(X)$ is a f.i. subgroup of $SL(2,\mathbb{Z})$.

Proof.
We note first that the stabilizer of a square-tiled surface X must be a subgroup of $SL(2,\mathbb{Z})$. Indeed the integral of 1 over any noncontractible simple closed curve or saddle connection is an element of $\mathbb{Z}\oplus i\mathbb{Z}$, and this must be preserved. A acts on these complex lengths via the linear action on $\mathbb{C}\simeq\mathbb{R}^2 \Longrightarrow A$ preserves the lattice, i.e. $A\in SL(2,\mathbb{Z})$.

Note ι that if $A\in SL(2,\mathbb{Z})$, $A\cdot X$ is also a square-tiled surface with the same number of squares: indeed, $SL(2,\mathbb{Z}) = \langle R=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, T=\begin{pmatrix}1&1\\0&1\end{pmatrix}\rangle$ and

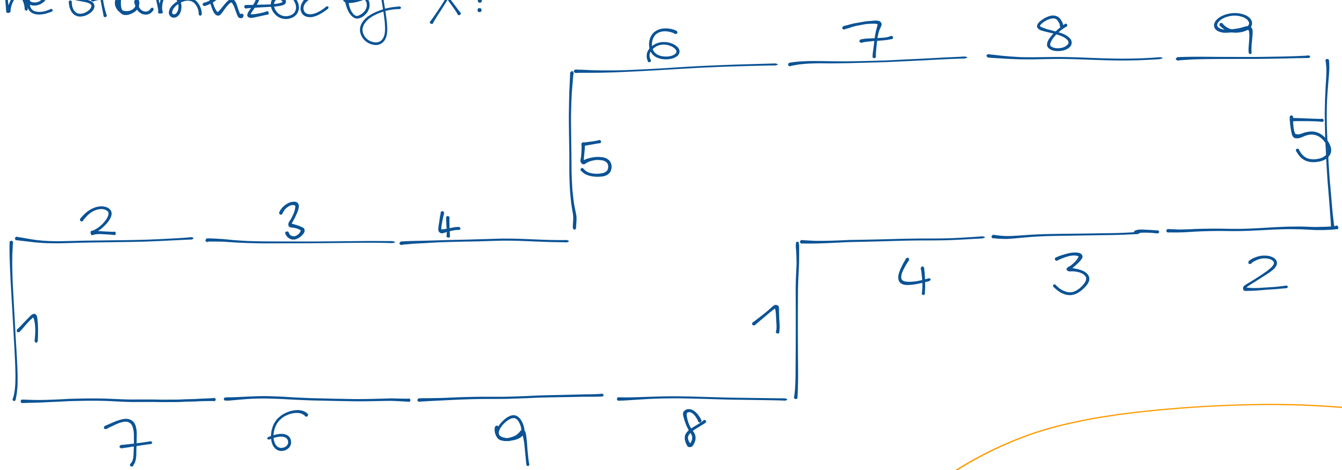
- R rotates the square by $\pi/2$, so $R\cdot X$ has the required property;
- T sends a unit square to a parallelogram with a horizonal side which is the horizontal side of the square and another side which is one of the diagonals of the square:



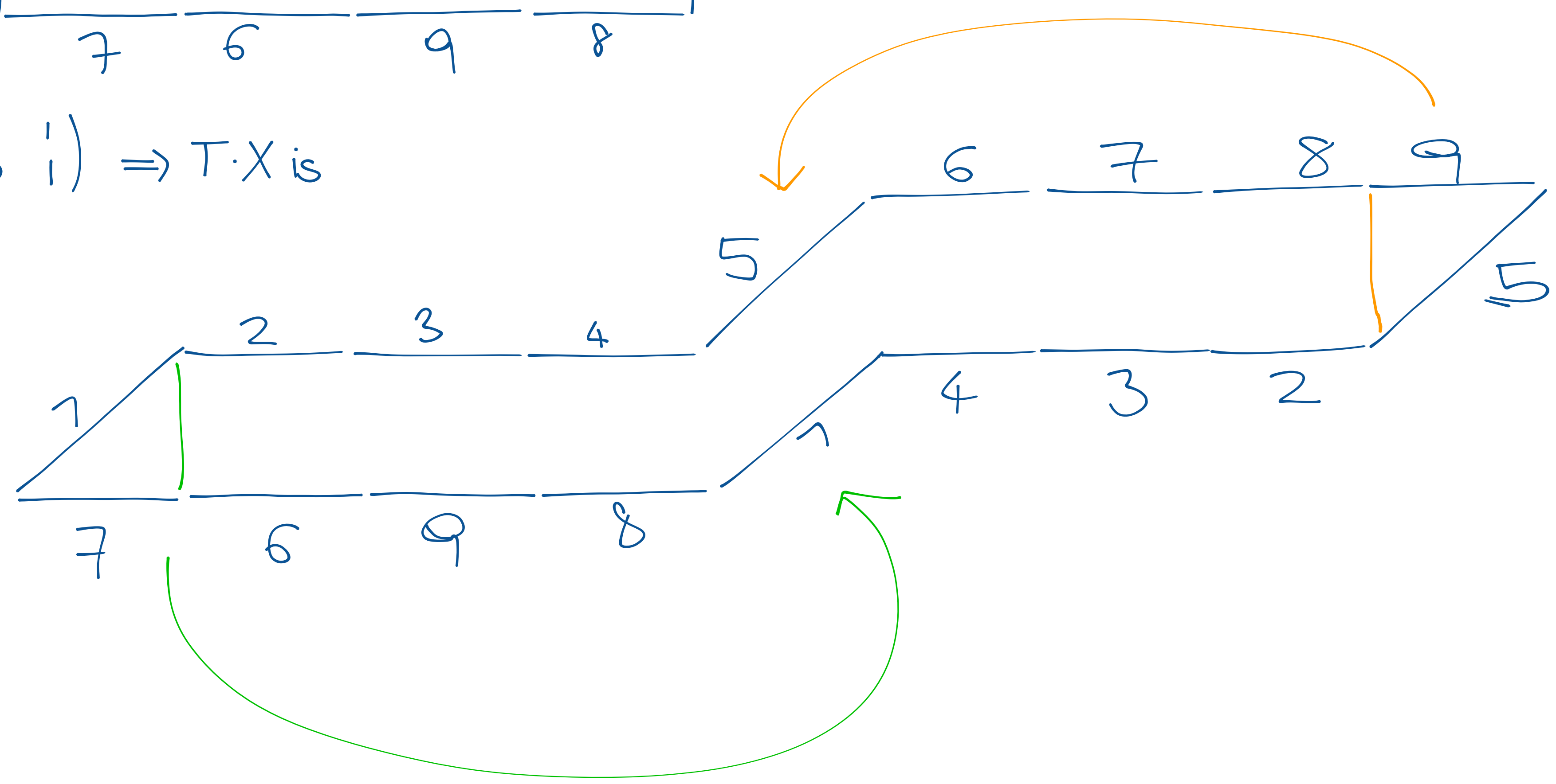
We can cut along the vertical diagonal and reglue to get again squares with sides identified.

Note that there are only finitely many - say N - square-tiled surfaces of a fixed number of squares. This implies that for any $A\in SL(2,\mathbb{Z})$, $A^N\in \mathrm{Stab}(X)$. Hence $[SL(2,\mathbb{Z}):\mathrm{Stab}(X)]\leq N$. $\square$

Ex. the stabilizer of X:

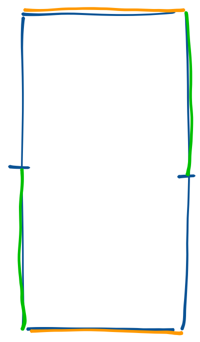


$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow T \cdot X$ is

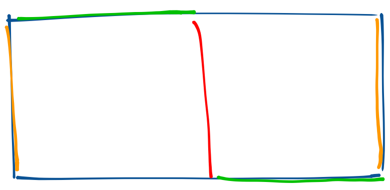
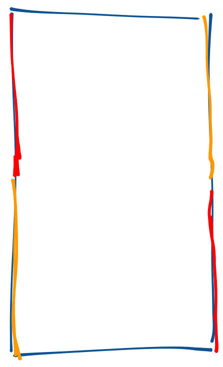


$\Rightarrow T \cdot X = X$

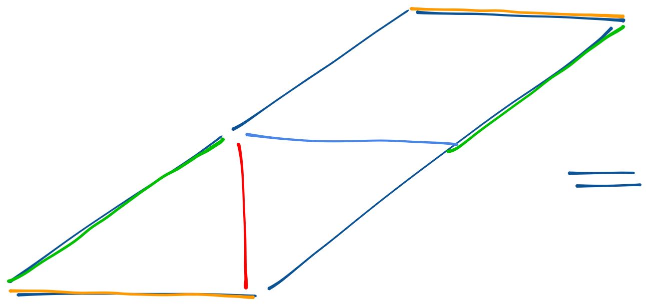
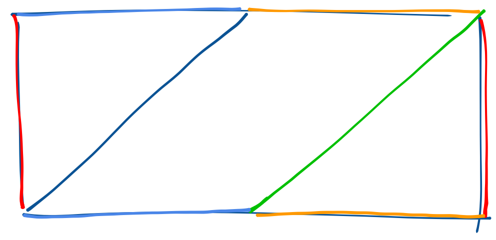
Similarly one can check (exercise) that $R \cdot X = X \Rightarrow$ since $SL(2,\mathbb{Z})$ is gen. by R and T, $\mathrm{Stab}(X) = SL(2,\mathbb{Z})$.

Ex.  $= Y$

Then $R \cdot Y$ is

 $=$  $= Y \Rightarrow R \in \mathrm{Stab}(Y)$.

Instead $T \cdot Y$ is

 $=$  $\neq Y$

because it has a vertical simple closed geodesic of length 1, while Y doesn't.

But one can check that $T^2 \cdot Y = Y$. So $[SL(2,\mathbb{Z}) : \mathrm{Stab}(Y)] = 2$.

The <u>Veech group</u> of a translation surface X, denoted $PSL(X)$, is the projection of the stabilizer $\mathrm{Stab}_{SL(2,\mathbb{R})}(X)$ to $PSL(2,\mathbb{R})$.

<u>Recall</u>: $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ with metric $\frac{|dz|}{\mathrm{Im} z}$ is the <u>hyperbolic plane</u>

One can show that $\mathrm{Isom}^+(\mathbb{H}^2) \simeq PSL(2,\mathbb{R})$ (acting as Möbius transf.: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$)

$\Rightarrow$ $PSL(X)$ acts on $\mathbb{H}^2$ by isometries and it is a discrete group (stabilizers are).
So the quotient ${}^{PSL(X)}\backslash \mathbb{H}^2$ is almost a surface (it can have cone pts) with a (sing.) hyp. metric and we can compute its area.

X is a <u>Veech surface</u> if the area of ${}_{PSL(X)}\backslash \mathbb{H}^2$ is finite.

<u>EX</u>: $X = (\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}, \omega_0)$ is a Veech surface

<u>Proof</u>: we know that $PSL(X) = PSL(2,\mathbb{Z})$ and that a fundamental domain for the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}^2$ is the set $\{z \in \mathbb{H}^2 \mid |\mathrm{Re}(z)| \leq 1/2,\ |z| \geq 1\} = F$.

We just need to check that the area of F is finite:

$$\mathrm{area}(F) = \int_F \frac{dx\,dy}{y^2} = \int_{-1/2}^{1/2} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy \right) dx = \int_{-1/2}^{1/2} \left[-\frac{1}{y} \right]_{\sqrt{1-x^2}}^{\infty} dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx < \infty$$ $\square$

<u>Rmk</u>: the metric "shrinks" as $\mathrm{Im} z \to \infty$ and "explodes" as $\mathrm{Im} z \to 0$ $\rightsquigarrow$ this is why F has finite area and for instance $\{z \mid |z| < 1\}$ doesn't.

<u>Consequence</u>: any squared-tiled surface X is a Veech surface, because the fact that the stabilizer is a f.i. subgroup implies that $\mathbb{H}^2/PSL(X)$ is a finite index cover of $\mathbb{H}^2/PSL(2,\mathbb{Z})$, so its area is a multiple of the area of $\mathbb{H}^2/PSL(2,\mathbb{Z})$.

Veech surfaces are interesting for many reasons. One is the following:

<u>Thm</u> (Smillie)
The $SL(2,\mathbb{R})$-orbit of a translation surface X is closed if and only if X is a Veech surface.

In general, orbits are very complicated and they are far from being closed:

<u>Thm</u> (Masur – Veech)
The $GL^+(2,\mathbb{R})$-orbit of a "generic" translation surface is dense in the connected component of the stratum which contains the surface.

So the "generic" orbit behaves very differently from the orbit of a Veech surface.

<u>Rmk</u>: "generic" means "almost every" with respect to a measure on each stratum which can be locally described by period coordinates.

The very varied types of orbits make it seem unlikely to have a classification of all orbits closures. So it is especially surprising that recently a concrete description has been proved.

An <u>affine invariant submanifold</u> is the image of a <u>proper</u> <u>immersion</u> f of an open connected mfold M to a stratum $\mathcal{H}(k)$ such that $\forall p \in M$ $\exists$ nbhood U s.t. $f(U)$ is given – in a nbhood of $f(p)$ – by linear equations in period coordinates with real coefficients and constant term 0.

$f^{-1}(\text{cpt})$ is cpt

differentiable with everywhere inj. derivative

<u>Thm</u> (Eskin–Mirzakhani–Mohammadi)
Any closed $GL^+(2,\mathbb{R})$–invariant set is a finite union of affine invariant submanifolds. In particular, orbit closures are affine invariant submanifolds