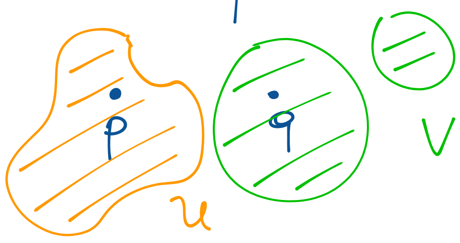
Basics in surface topology

Surface = (second countable, Hausdorff) top. space locally homeomorphic to $\mathbb{R}^2$

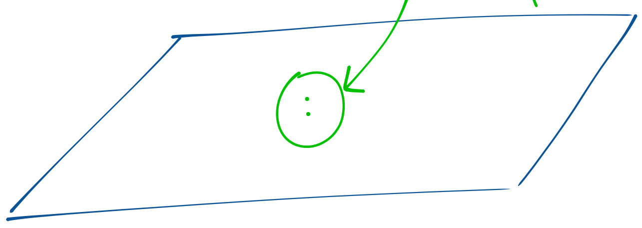
- second countable: w/ a countable base for the top. ($\exists \{U_i\}_{i\in\mathbb{N}}$ s.t. any open set is a union of U_i's)
  EX. $\mathbb{R}^n$: take balls of rational radius of center in $\mathbb{Q}^n$
- Hausdorff: $\forall p \neq q$ $\exists$ open nbhoods U of p and V of q s.t. $U \cap V = \phi$



- locally homeomorphic to $\mathbb{R}^2$: $\forall p \in S$ $\exists$ open nbhood U homeomorphic to an open set V of $\mathbb{R}^2$
  $\rightarrow$ EX. $\mathbb{R}^n$, or more generally any space whose topology is induced by a metric

Why these hypotheses?

① $X = \mathbb{R}^2 \times \{a,b\} / \sim$ where $(x,a) \sim (x,b)$ if $x \neq 0$; $\pi : \mathbb{R}^2 \times \{a,b\} \to X$ projection



Rmk: loc. homeo to $\mathbb{R}^2$ (for $[0,a]$, $\pi(\mathbb{R}^2 \times \{a\})$ is an open set homeo to $\mathbb{R}^2$)
but not Hausdorff: no disjoint nbhoods of $[0,a]$ and $[0,b]$

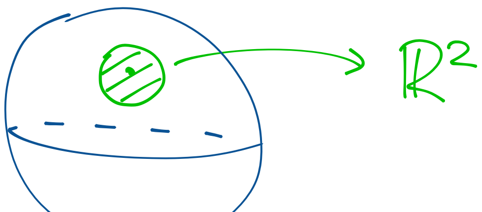
② Prüfer surface: $\mathbb{R}^2 \times \mathbb{R} \sqcup H / \sim$, $H = \mathbb{R} \times \mathbb{R}^+$

$$(x,y,a) \sim (a+yx, y) \quad \forall y>0$$

Can show: loc. homeo to $\mathbb{R}^2$ but not second countable

Basic ex of surfaces: • $\mathbb{R}^2$ (obviously)

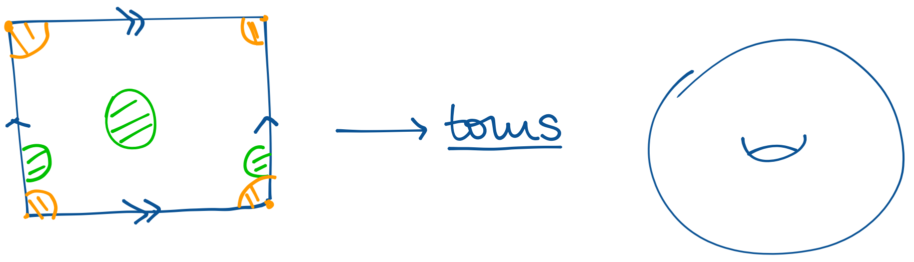
• S^2 ($\forall p \neq N \rightsquigarrow$ use stereographic proj. from N of $S^2 \setminus N$ to $\mathbb{R}^2$
For $N \longrightarrow$ proj. from S)



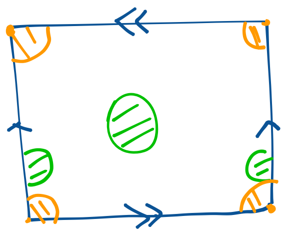
$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

$p_N(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ $(z \neq 1)$

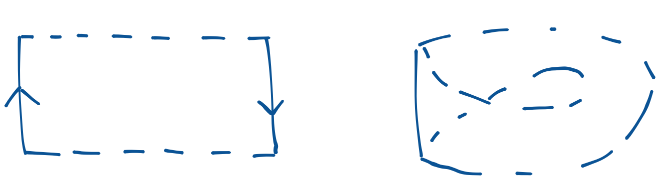
• gluings of polygons



$\longrightarrow$ torus



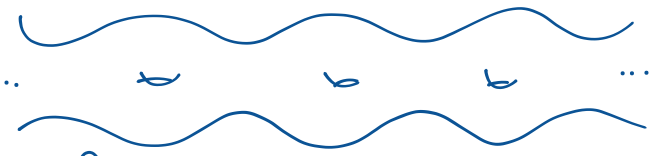
$\longrightarrow$ Klein bottle $\rightsquigarrow$ non-orientable surface
it contains a Möbius band



We will exclude this: consider orientable surfaces.

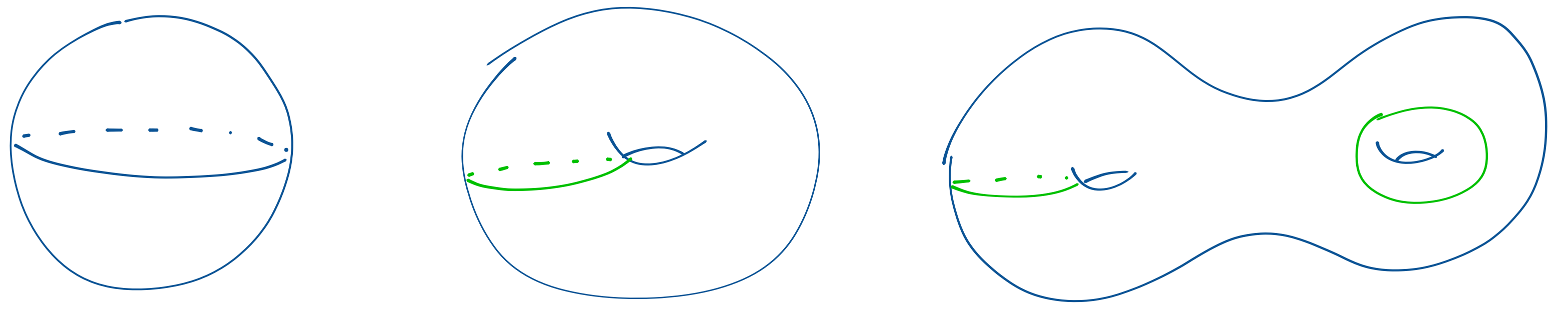
Also: assume connected.

S surface is closed if cpt.

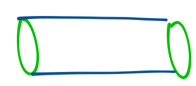
EX. S^2 is, $\mathbb{R}^2$ isn't, but neither is ...  ...

The genus of a surface is the max # of homotopically non-trivial simple closed curves $\gamma_1, \dots, \gamma_k$ pairwise disjoint s.t. $S \setminus \bigcup \gamma_k$ is connected.

EX:



every curve separates (Jordan curve theorem) $\Rightarrow$ genus $= 0$



$\uparrow$ any curve separates this $\Rightarrow g = 1$

genus 2

Idea: genus $=$ # "holes"

Classification of cpt surfaces

Up to homeo, a cpt surface is det by its genus.

Rmk: we can get them all as quotients of polygons

How do we see this?

$\longrightarrow$ Euler characteristic: use a cell dec. of S, i.e.: start w/ finitely many pts $p_1, \dots, p_v$, attach segm. $l_1, \dots, l_e$ ($\simeq [0,1]$) by identify each starting and endpt of any segment to the pts, and disks $D_1, \dots, D_f$ identifying the ∂ circles w/ the one-skeleton

1-skeleton

More general definition:

X Hausdorff is a (finite) CW-cpx if it is obtained as follows: start w/ X_0 a finite collection of pts and form inductively X_n (n skeleton) by attaching finitely many n-cells (n-dim closed balls) to X_{n-1}: $\{C_\alpha\}_{\alpha \in A}$ n-dim closed balls, finite

$f_\alpha : \partial C_\alpha \longrightarrow X_{n-1}$ continuous, then

attaching maps

$$X_n = X \sqcup_\alpha C_\alpha \Big/ f_\alpha(x) \sim x$$

We assume: we have finitely many cells in total.

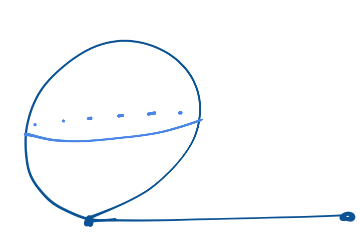
For such X, we can define the Euler characteristic as

$$\chi(X) := \sum_{i \geq 0} (-1)^i \, \# \, 0\text{-cells}.$$

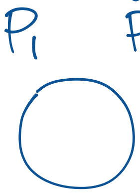
Thm: It is indep. on the decomposition.

EX. $X = X_0 = \cdot \;\dot{}\; \cdot \Rightarrow \chi(X) = 3$

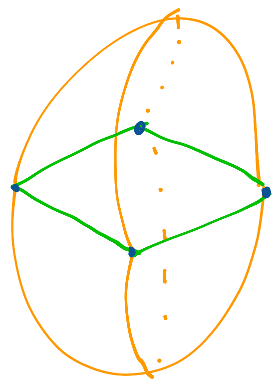
EX. $X_0 = \overset{A}{\cdot} \quad \overset{B}{\cdot}$



Attach $\underset{P_1}{\bullet}\!\!\longrightarrow\!\!\underset{P_2}{\bullet}$ by sending P_1 to A and P_2 to B

Attach  by sending $\partial \rightarrow A = P_1$

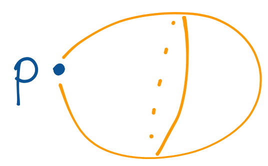
Ex.



$\longrightarrow$ sphere

$\chi = 4-4+2 = 2$

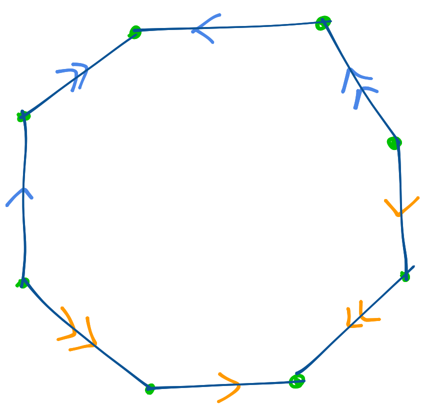
Note: we can get the sphere also as



Attaching map:
$f: \partial D^2 \longrightarrow \{p\}$

We get the same Euler characteristic: $\chi = 1-0+1=2$.

Ex.



1 pt, 4 edges, 1 face: $\chi(S) = 1-4+1 = -2$

Fact: $\chi(S_g) = 2-2g$

surface of genus g

Note: we could take this as a definition of genus for cpt, conn, or surfaces.

Exercise: show that one can get any closed surface by gluing polygons