The straight line flow

Given a space X, a <u>flow</u> on X is a map

$$\varphi : X \times \mathbb{R} \longrightarrow X$$
$$(x,t) \longmapsto \boxed{\varphi_t(x)}$$ flow at time t of the pt x

such that

(1) $\varphi_0(x) = x \quad \forall x \in X$

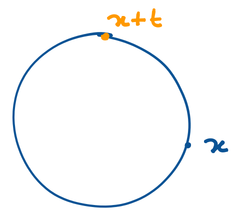
(2) $\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x) \quad \forall x \in X, \forall t,s \in \mathbb{R}$.

group action of $\mathbb{R}$ on X

<u>Rmk</u>: depending on which structure X has, we can ask more properties of the flow. For ex. if X is a top. space, we can ask φ to be continuous.

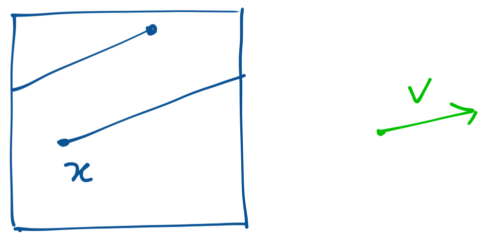
<u>Ex</u>

(1) $S^1 = \mathbb{R}/\mathbb{Z} \rightsquigarrow \varphi_t([x]) = [x+t]$



(2) $X = \mathbb{R}^n$, $A \in Mat(n \times n, \mathbb{R}) \rightsquigarrow \varphi_t(x) = \exp(tA) \cdot x$

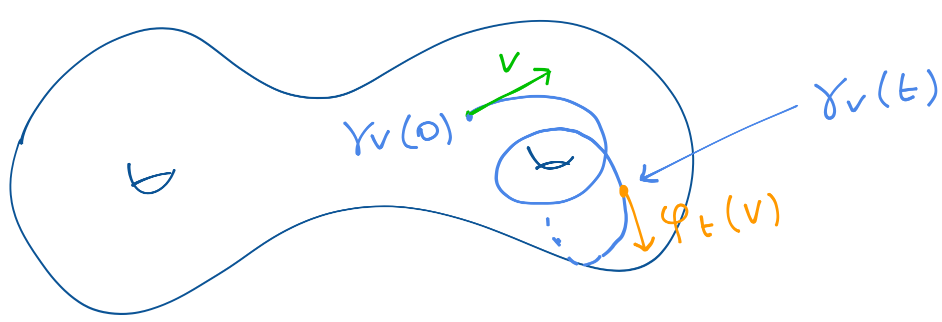
(3) $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $v \in \mathbb{R}^2 \rightsquigarrow \varphi_t^v([x]) = [x+tv]$



<u>Rmk</u>: the orbit of a pt is the geodesic starting at the pt and going in direction v

(4) geodesic flow of a Riemannian mfld M: flow on $\underbrace{T^1M}_{\text{unit tg bundle}}$, given by

$$T^1M \times \mathbb{R} \longrightarrow T^1M$$
$$(v,t) \longmapsto \left.\frac{d\gamma_v(s)}{ds}\right|_{s=t}$$ where γ_v = geodesic determined by v



The flow we want to consider is the same as in (3), but for general transl. surfaces. Note that given X translation surface defined by gluing of polygons we can talk about vectors in a given direction based at any nonsingular pt: any gluing map preserves this. The same holds if we think of the singular flat structure: transitions are translations $\rightsquigarrow$ can transport the direction.

Moreover, starting at p non singular and given a vector $v \in \mathbb{R}^2$, we know what it means to go straight in direction v, until we hit a singularity. Thinking of the flat structure, it means following the geodesic starting from the p of tg vector at p given by v. When we hit a singularity, we do not know how to proceed. So we consider a flow on a subset of a translation surface X.

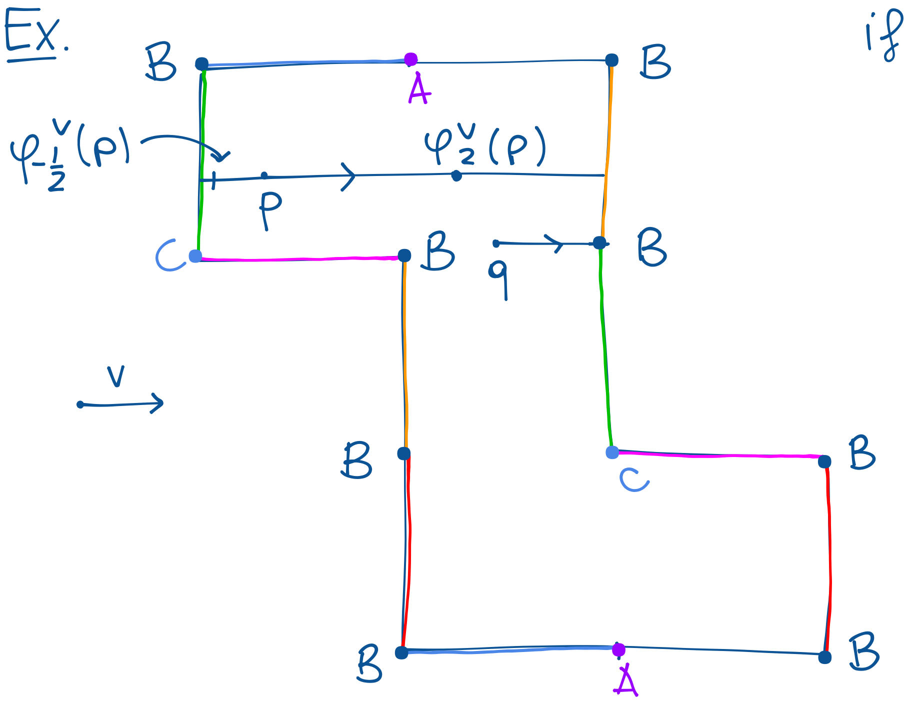
$\Sigma = \{\text{singularities of } X\}$. Fix $v \in \mathbb{R}^2$.

For $p \in X \setminus \Sigma$, define $\varphi_t^v(x) :=$ pt obtained travelling in direction v tracing a path of $\boxed{\text{signed length } t}$

for as long as we don't hit a singularity

if $t<0 \rightsquigarrow$ go in dir $-v$ for time $|t|$

Ex.



$\chi = 3-6+1 = -2 \Rightarrow g = 2$

Rmk: A, C not singular

Define $B_t := \{p \mid \varphi_s^v(p) \text{ hits a singularity for some } |s|<t\}$.

$\Rightarrow$ we have a well-defined flow on $X \setminus \bigcup_{t>0} B_t$. Set $B := \bigcup_{t>0} B_t$.

Rmk: the set B has measure 0. Indeed, $\bigcup_{t>0} B_t = \bigcup_{n\in\mathbb{N}} B_n$ and $\forall n$, B_n is a union of finitely many segments, hence B_n has measure zero and a countable union of measure zero sets has measure zero.

Note though that the union could be dense in X!

The straight line flow is the flow $\varphi^v : (X\setminus B) \times \mathbb{R} \longrightarrow X \setminus B$.

We want to understand properties of this flow.

For $p \in X \setminus B$, the image $\varphi^v_{\mathbb{R}}(p)$ is the orbit of p under the flow. The simplest orbits are those that are periodic: the orbit of p is periodic if $\exists T>0$ s.t.

$$\varphi^v_T(p) = p.$$

By the properties of a flow, this is equivalent to

$$\varphi^v_{T+t}(p) = \varphi^v_t(p) \text{ for all } t \in \mathbb{R}.$$

Rmk: periodic orbits of the straight line flow are closed geodesics in one direction. If the translation surface comes from a billiard, they correspond to periodic billiard trajectories.

EX: $T =$ unit square torus

$0 \neq v = (v_1, v_2)$ for $v_1 = 0$ or $v_1 \neq 0$ and $\frac{v_2}{v_1} \in \mathbb{Q}$ (v has rational slope)

$\Rightarrow \varphi^v$ is periodic for every $p = [x,y] \in \mathbb{R}^2/\mathbb{Z}^2$

- $v_1 = 0 \Rightarrow T = \frac{1}{v_2} \rightsquigarrow \varphi^v_T(p) = [x, y + \underset{\overset{\shortparallel}{1}}{T v_2}] = [x,y] = p$
- $v_1 \neq 0 \Rightarrow \frac{v_2}{v_1} = \frac{p}{q}$, $p \in \mathbb{Z}, q \in \mathbb{N}$

$$T := \frac{q}{v_1}$$

$$\frac{q}{v_1} \cdot \frac{p}{q} \cdot v_1 \in \mathbb{Z}$$

Then $\varphi^v_T(p) = [x + \overset{\in\mathbb{Z}}{\frac{q}{\cancel{v_1}}\cancel{v_1}}, y + \frac{q}{v_1} v_2] = p$.

What if v has irrational slope?

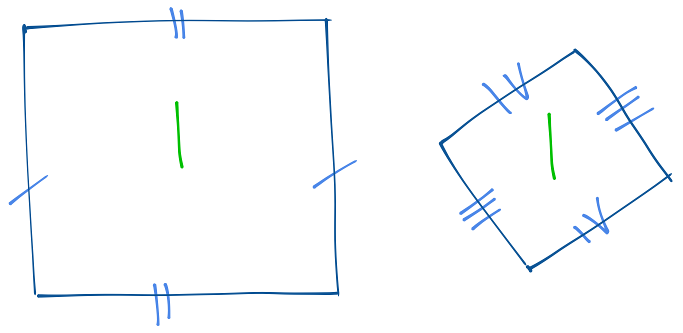
Then $\varphi_t^v(p)$ is injective (as fct of t): otherwise, if $\exists\, t \neq s$ s.t. $\varphi_t^v(p) = \varphi_s^v(p)$ we have $[x+tv_1, y+tv_2] = [x+sv_1, y+sv_2]$, i.e. $\exists\, n, m \in \mathbb{Z}$ s.t.

$$\begin{cases} x+tv_1 = x+sv_1+n \\ y+tv_2 = y+sv_2+m \end{cases} \Rightarrow \begin{cases} v_1 = \dfrac{n}{t-s} \\ v_2 = \dfrac{m}{t-s} \end{cases} \Rightarrow \frac{v_1}{v_2} \in \mathbb{Q} \quad \lightning$$

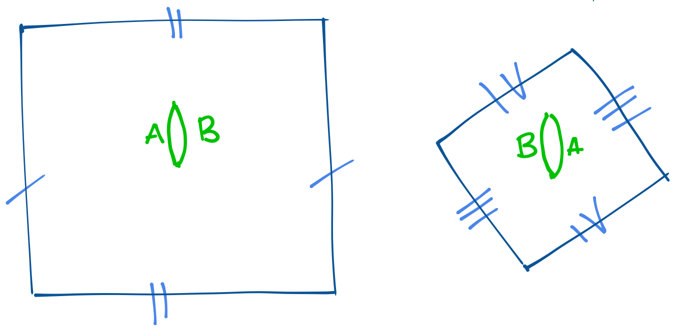
So since the torus is cpt, the orbit of any pt has an accumulation point.
Actually one can show that the orbit is dense

the closure is T, or equivalently $\forall q \in T$, $\forall \varepsilon > 0$ $\exists\, t \in \mathbb{R}$ s.t. $\varphi_t^v(p)$ is at distance at most ε from q.

Ex: the slit torus construction. Start with two tori and choose parallel segments of the same length, one in each torus:



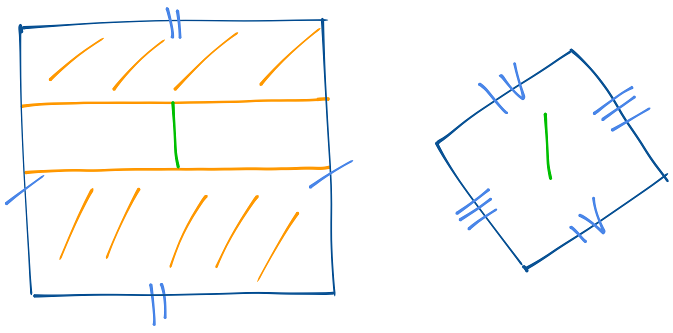
Cut along the slit in both tori and to get two tori with boundary given by two segments. Topological picture:



Glue the two boundary components as indicated (the right side of a slit is glued to the left side of the other slit). We get a genus two translation surface with two cone pts of angles 4π each.

Exercise: describe the surface as gluing of polygons.

Supp. now we have such a surface, where the square to the right has an irrational slope. Look at the horizontal straight line flow:



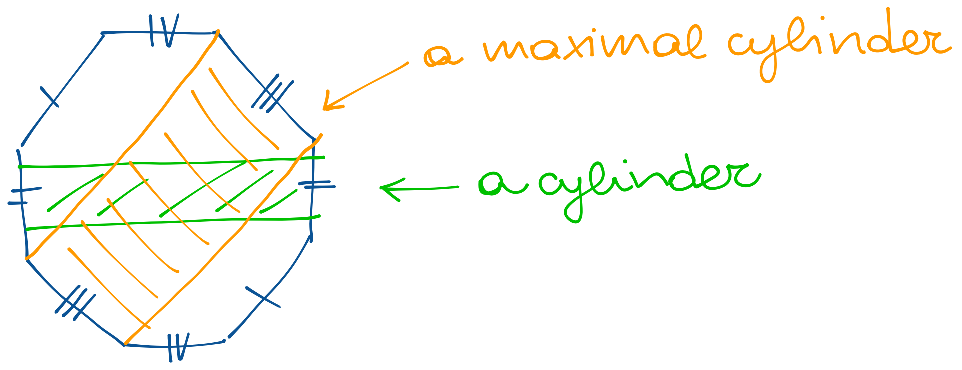
For any pt in the orange part, the orbit is periodic. For all other pts, the orbit is dense in the white part.

A flow is minimal if every orbit is dense.

We just saw: the flow on the unit square torus for a direction with irrational slope is minimal. On the slit torus "with irrational slope", the horizontal flow is not minimal. To see what happens in general, we need first a definition.

A cylinder on a translation surface X is an isometrically embedded Euclidean cylinder. A cylinder is maximal if its boundary is a union of saddle connections.

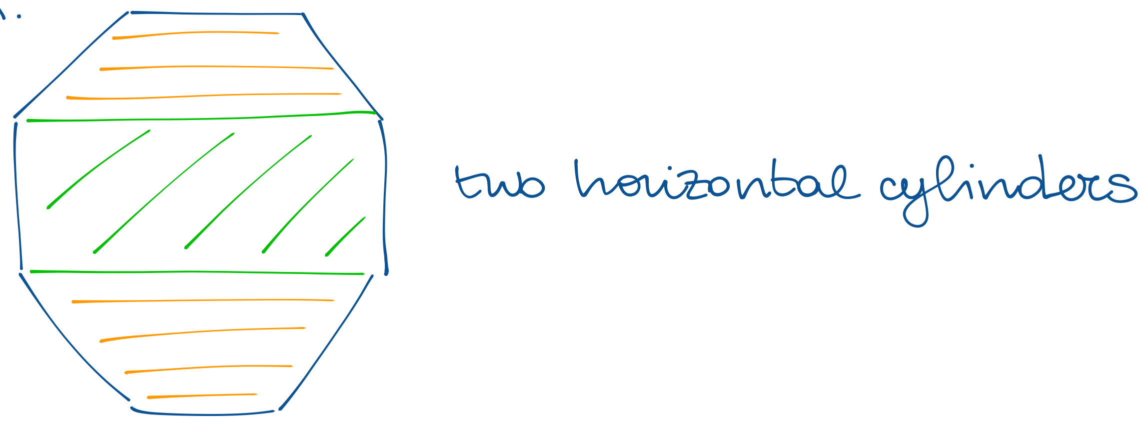
Ex.



Given a cylinder, its direction is the direction of its boundary curves / saddle connections. If a cylinder is isometric to $\mathbb{R}/c\mathbb{Z} \times [0,h]$, we call h its height and c its circumference.

A surface is periodic in one direction if it is the union of maximal cylinders in that direction.

Ex.



Rmk: if X is periodic in a direction v, every orbit of φ^v is periodic.

Cylinders are useful to decompose the surface into "independent" parts with respect to the straight line flow:

Thm (Katok–Zemljakov)

Given a translation surface X of genus ≥ 2 and a direction v, after removing all saddle connections in the direction v, each connected component is either a cylinder or it has a minimal flow in direction v.

Rmk: this is exactly what we observed for the horizontal flow for the slit torus surface.

As a consequence we have:

Cor.

For any translation surface of genus ≥ 2, the straight line flow is minimal in all but countably many directions.

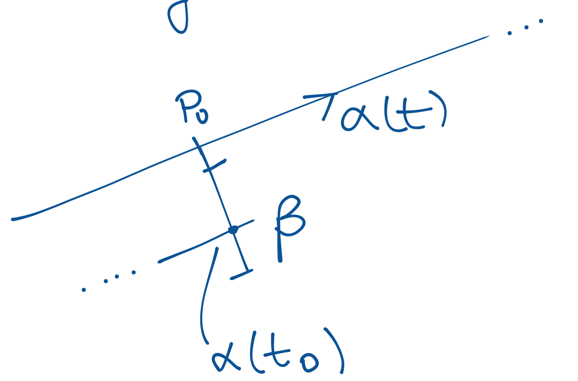
Proof. Since there are finitely many singularities, there are at most countably many saddle connections. In any direction without saddle connections the flow is minimal by the previous result. $\square$

We will prove the Katok–Zemljakov's theorem. The followin lemma will be useful. For this, we will consider the flow as defined on the entire surface, by saying that whenever we hit a singularity the flow stays there forever.

Lemma

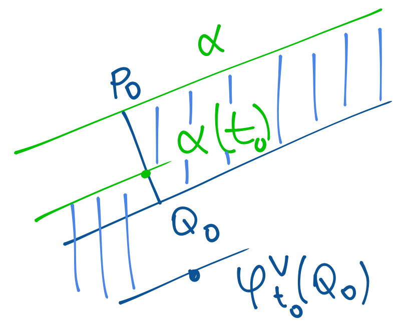
Suppose an orbit $\alpha(t)=\varphi^V_t(P_0)$ never hits a singularity for $t\geq 0$. Let β be a segment with P_0 as an endpt and perpendicular to $\alpha(t)$. Then $\exists t_0>0$ s.t. $\alpha(t_0)\in\beta$.

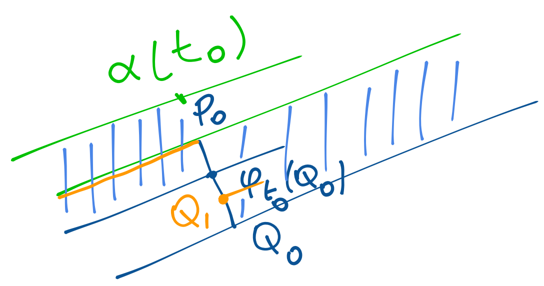
Schematically:



Proof. Note that there are only finitely many pts $P\in\beta$ s.t. $\varphi^V_{t>0}(P)$ hits a singularity before hitting β again. So up to shrinking β to β' w/ endpts P_0, Q_0, we can assume $\varphi^V_{t>0}(P)$ does not hit a singularity before going back to β $\forall P\in\beta'$.

Look at $\varphi_{[0,t]}(\beta')\longrightarrow$ embedded rectangle of area $t\cdot\ell(\beta')$ for as long as it does not hit β' again. Since the area of the surface is finite, $\exists$ a first $t_0>0$: $\varphi_{t_0}(\beta')\cap\mathrm{Int}(\beta')\neq\phi\implies$ either $\alpha(t_0)\in\beta'$ and we are done, or $\varphi^V_{t_0}(Q_0)\in\beta'$.





In the second case $\exists Q_1\in(\beta')^\circ$ s.t. $\varphi^V_{t_1}(Q_1)=P_0$ for some $t_1>0$. Consider then β'' with endpts P_0 and Q_1 and repeat the argument with β'' to get that $\exists t_1>0$ s.t. $\varphi_{t_1}(\beta'')\cap(\beta'')^\circ\neq\phi$. If $\alpha(t_1)\in(\beta'')^\circ$, we are done. We will show that the other possibility does not happen.

Note first that if at some time t, $\varphi^V_t(\beta'')\cap\mathrm{Int}(\beta'')\neq\phi$, then for the same t $\varphi^V_t(\beta')\cap\mathrm{Int}(\beta')\neq\phi$. So the minimality of t_0 implies that $t_1\geq t_0$, i.e. $\exists t_2\geq 0$: $t_1=t_0+t_2$. Supp. $\alpha(t_1)\notin\mathrm{Int}(\beta'')$. As before, this implies that $\exists Q_2\in\mathrm{Int}(\beta'')$ s.t. $\varphi^V_{t_1}(Q_2)=P_0$.

$$\varphi^V_{t_1}(Q_2) = \varphi^V_{t_0}(\varphi^V_{t_2}(Q_2))$$

As $\varphi^V_{t_0}(Q_1)=P_0$, we get $\varphi^V_{t_2}(Q_2)=Q_1$. So at time t_2

$$\varphi^V_{t_2}(\beta'')\cap\mathrm{Int}(\beta'')\neq\phi$$

and by the minimality of t_1, $t_2\geq t_1=t_2+t_0$ ↯

□

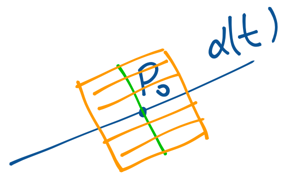
Proof of thm. Consider φ_t^v.
Cut X along the saddle connections in the direction of the flow and let P be a pt in a conn. component of the cut surface.

Case 1: the orbit of P is periodic.
We show then that the component containing P is a maximal cylinder. Indeed for some small ε, the ε-nbhood of the orbit of P is a cylinder. Enlarge the cylinder for as long as we can. The only obstruction is if we hit a sing, so we get a cylinder containing the orbit of P w/ singularities on each boundary. So its boundary is a union of saddle connections in direction v, i.e. it is a max cylinder and a conn. component of the cut surface.

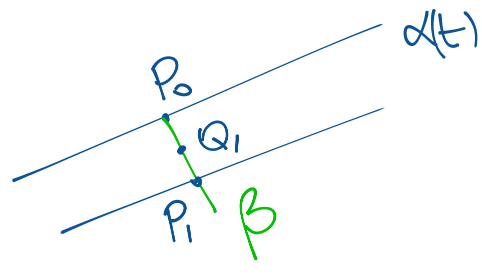
Case 2: the orbit of P is not periodic. As a consequence of the previous case, no orbit is periodic. Claim: the flow is minimal in this component. Suppose $\exists Q$ in the connected component whose orbit is not dense. Call the conn. component Y.

Let A be the closure of the orbit of Q. Note that since the flow is continuous, A is φ_t-invariant. Furthermore $A \neq Y$, because the orbit of Q is not dense. Thus $\partial A \neq \phi$ and we note that any pt in ∂A has a small segment in direction ε which is still in A: if $\varphi_{t_k}^v(Q) \longrightarrow P \in \partial A$, for ε small enough s.t. $d(\varphi_t^v(Q), \varphi_{t+\varepsilon}^v(Q)) = \varepsilon \ \forall t$, $\varphi_{t_k+\varepsilon}^v(Q) \longrightarrow$ "$P+\varepsilon v$".

Let $\alpha(t) := \varphi_t^v(P_0)$ be a non-constant orbit in $\partial A \setminus \partial Y$. We'll show that $\exists$ small nbhood of P_0 contained in A, contradicting the fact that $P_0 \in \partial A$. Note that it is enough to show that $\exists$ small interval $\perp \alpha(t)$ with P_0 in its interior and contained in A. Then by flowing slightly backwards and forwards we get the neighborhood.



Let β be an interval orthogonal to $\alpha(t)$, starting at P_0 (this will give us half the interval, but we can just repeat the construction on the other side).
Using the lemma, we know that $\alpha(t)$ hits β again at some P_1. If $[P_0, P_1] \subseteq A$, we are done, so suppose not, i.e. $\exists Q_1 \in [P_0, P_1] \setminus A$:



Let I_1 be the largest open interval in $\beta \setminus A$ containing Q_1 – it exists since A is closed. Let P_2 be the endpt of I_1 between P_0 and Q_1. Then $P_2 \in A$ and the orbit of P_2 is a saddle connection: if not, by the lemma it hits I_1 again, which implies (since A is flow-invariant) that $A \cap I_1 \neq \phi$, impossible by construction of I_1.
But we assumed that there are no saddle connections in direction of v in Y ↯

$\square$

(Unique) ergodicity

Given a space X w/ a probability measure m, a flow φ is <u>measure preserving</u> if $\forall U \subseteq X$ measurable set, $\varphi(U)$ is measurable and

$$m(U) = m(\varphi(U)).$$

A measure preserving flow φ on (X, m) is <u>ergodic</u> if $\forall U \subseteq X$ measurable s.t. U is φ-invariant (i.e. $\varphi_t(U) \subseteq U \ \forall t$), either $m(U) = 0$ or $m(U) = 1$.

In words: all invariant subsets have measure zero or full measure.
In our case, the measure is the one induced by the Lebesgue measure μ, renormalized to have total mass one:

$$m(U) := \frac{\mu(U)}{\underbrace{\boxed{\mu(X)}}_{\text{area of } X}}.$$

In the example of the torus, one can show that φ^v is ergodic if v has irrational slope, not ergodic otherwise.

A way to think of what ergodicity means is given by the following result:
<u>Thm</u> (Birkhoff Ergodic Theorem)
Let Y be a space with a probability measure m and φ a measure preserving ergodic flow on Y. Let $f \in L^1(Y, m)$. Then for m-a.e. $y \in Y$

$$\lim_{T\to\infty} \underbrace{\frac{1}{T}\int_0^T f(\phi_t(y))\,dt}_{\text{"average" of } f \text{ on } \phi_{[0,T]}(y)} = \underbrace{\int_Y f\,dm}_{\text{"average" of } f \text{ on } Y}.$$

This tells us in particular that the proportion of time the orbit of an ergodic flow spends in a subset tends to the measure of the set $\Rightarrow$ the flow goes everywhere "evenly". Precisely, for $A \subseteq Y$ open set, we can apply the thm to $f = \chi_A$ and we get that for a.e. $y \in Y$

$$\lim_{T\to\infty} \frac{1}{T}\underbrace{\int_0^T \chi_A(\varphi_t(y))\,dt}_{\mu\{t\in[0,T] \mid \varphi_t(y)\in A\}} = \int_Y \chi_A\,dm = m(A)$$

Lebesgue measure
(std measure on $\mathbb{R}$)

A flow φ on a space Y is <u>uniquely ergodic</u> if $\exists!$ probability measure m s.t. φ is m-invariant.
<u>Lemma</u>
Unique ergodicity implies ergodicity.

<u>Proof</u>: suppose not. Then $\exists A \subseteq Y$ with $m(A) \in (0,1)$ and A is φ-invariant. Then we can consider the measure $\bar{m}$ given by, $\forall$ measurable set B:

$$\bar{m}(B) := \frac{m(A\cap B)}{m(A)}.$$

Note that since A is invariant, $\varphi_t(A) \subseteq A \ \forall t$. But φ_t is also measure preserving, so $m(\varphi_t(A)) = m(A)$ and hence $m(A \setminus \varphi_t(A)) = 0$.

This is a probability measure and it is φ-invariant:

$$\bar{m}(\varphi_t(B)) = \frac{m(A\cap\varphi_t(B))}{m(A)} = \frac{m(\varphi_t(A)\cap\varphi_t(B))}{m(A)} + \underbrace{\boxed{\frac{m((A\setminus\varphi_t(A))\cap\varphi_t(B))}{m(A)}}}_{=0 \text{ by the rmk above}}$$

$$= \frac{m(\varphi_t(A\cap B))}{m(A)} = \frac{m(A\cap B)}{m(A)} = \bar{m}(B).$$

Moreover, $m\neq\bar{m}$: $m(Y\setminus A)\neq 0$ but $\bar{m}(Y\setminus A)=0$. ↯

$\square$

Rmk: Ergodic $\not\Rightarrow$ uniquely ergodic (there are ex for φ^v).

Ergodic & minimal are independent notions for flows in general:

(1) the geodesic flow on the unit tangent bundle of a hyperbolic surface can be shown to be ergodic, but it has many periodic orbits – all those corresponding to closed geodesics on the surface.

(2) $\exists$ translation surfaces whose flow is minimal but not ergodic. Actually one can show:

Thm (Cheung–Masur)
Any translation surface of genus 2 which is not a Veech surface has a direction in which the flow is minimal and not ergodic.

One of the very interesting ideas is that there is an interplay between the straight line flow and the $GL^+(2,\mathbb{R})$-action. Note that:

(1) studying φ^v on a surface X (v unit vect), is the same as studying the vertical line flow on $R_\theta X$, for the appropriate θ;

(2) an orbit segment of the vertical flow on X of length L corresponds to an orbit segment of length $e^{-s}L$ for the vertical flow on $G_s\cdot X$ (where $G_s=\begin{pmatrix} e^s & 0\\ 0 & e^{-s}\end{pmatrix}$). Using this process, called renormalization, we can study the flow on X by studying a different but related dynamical system.

This idea has been used to show a criterion for unique ergodicity.

The positive G_t-orbit $\{G_t\cdot X \mid t\geq 0\}\subseteq\mathcal{H}(k)$ is recurrent if $\exists K\subseteq\mathcal{H}(k)$ cpt s.t. $\forall t\gg 0$ $G_t\cdot X\in K$.

Ex: if X = unit square, its orbit is not recurrent. In general, if a translation surface X has a vertical saddle connection, then also $G_t\cdot X$ has a vertical saddle connection, whose length goes to zero. So by Masur's criterion, $G_t\cdot X$ is unbounded.

Using this notion, we get a sufficient condition for unique ergodicity of the flow:

Thm (Masur's unique ergodicity criterion)
If $\{G_t\cdot X\}_t$ is recurrent, the vertical straight line flow is uniquely ergodic.

Note that this shows also how sometimes to study properties of a fixed translation surface it is useful to study the orbit of the surface under the action of (a subgroup of) $SL(2,\mathbb{R})$.

We also have the following:

Thm (Kerckhoff – Masur – Smillie)
For every X and for almost all $\theta \in [0,2\pi)$, $\{G_t(R_\theta X)\}_t$ is recurrent.

Combining the two results, we get that the straight line flow in almost every direction is uniquely ergodic. So from the measure-theoretic viewpoint, there are very few directions in which there is a periodic orbit. On the other hand:

Thm (Masur)
For any translation surface, the set of directions in which the straight line flow has a periodic orbit is dense in $[0,2\pi)$.

$\Updownarrow$

$\exists$ (maximal) cylinder

Rmk: for the square torus, the directions in which the flow is periodic are $(\mathbb{Q}\pi \cap [0,2\pi))$, which is dense and of measure zero. In all other directions the flow is uniquely ergodic. This is a strong dichotomy, which does not hold in general. Though it holds for Veech surfaces:

Thm (Veech dichotomy)
Let X be a Veech surface. The straight line flow is either periodic or uniquely ergodic. If the genus is at least two, the directions in which the flow is periodic are those in which there is a saddle connection.

For (a sketch of) the proof of the dichotomy we'll use a result relating properties of the vertical line flow on a surface to the properties of its stabilizer in $SL(2,\mathbb{R})$.

Recall that a cylinder had a height h and a circumference c. The modulus of the cylinder is the quotient $\frac{h}{c}$.

Prop.
$\exists s \neq 0$ s.t. $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in \mathrm{Stab}_{SL(2,\mathbb{R})}(X) \iff$ X is the union of vertical cylinders of moduli m_i s.t. s is an integer multiple of m_i for all i.

Proof.
$\boxed{\Leftarrow}$ Given a vertical cylinder of height h and circumference c, one can check that it is stabilized by $\begin{pmatrix} 1 & 0 \\ \frac{h}{c}k & 1 \end{pmatrix}$ for all $k \in \mathbb{N}$. So all cylinders are stabilized by $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and we get that X is stabilized by $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ too.

$\boxed{\Rightarrow}$ We will use a fact about the stabilizer of a translation surface for which we need to talk about affine maps of translation surfaces.
$\phi: X \to X$ is an affine map if it is a bijection between singularities and in local charts away from singularities it is an affine map.
Fact: $\mathrm{Stab}(X) = \{$linear parts of affine or. pres. homeomorphisms of $X\}$.

Using this, let ϕ be an affine map having $\begin{pmatrix}1&0\\s&1\end{pmatrix}$ as linear part. Let $\{L_1,\dots,L_k\}$ be the set of all vertical geodesics starting at singularities. Since $\begin{pmatrix}1&0\\s&1\end{pmatrix}$ preserves the vertical direction, ψ permutes Σ and $\{L_1,\dots,L_k\}$. So $\exists N>0$ s.t. $\phi=\psi^N$ fixes all singularities and all L_i's. On each L_i, ψ acts as a translation, but since it fixes the singularities, it must be the identity.

Next we show that all L_i are saddle connections. Supp. one isn't; then it is dense in a subsurface. But since $\psi|_{L_i}=id_{L_i}$ this would mean that ψ is the identity (on the subsurface), but its linear part is $\begin{pmatrix}1&0\\sN&1\end{pmatrix}\neq Id$, a contradiction.

Now we know that <u>all</u> vertical geodesics starting from singularities are saddle connections. If we cut along all of them, we get surfaces without singularities, which thus must be cylinders.

$\square$

<u>Sketch of proof of the Veech dichotomy</u>:

Consider $\{G_t\cdot X\}_t$. If it is recurrent, by Masur's unique ergodicity criterion the vertical straight line flow is uniquely ergodic. So we need to show that if $\{G_t\cdot X\}_t$ is <u>not</u> recurrent, the vertical line flow is periodic.

If $\{G_t\cdot X\}_t$ leaves every cpt (i.e. it is not recurrent), then by Masur's compactness criterion the length of the shortest saddle connection of $G_t X$ goes to 0 as $t\to\infty$.

<u>Claim</u>: this implies that $[G_t]\longrightarrow\infty$ (leaves every cpt) in $\boxed{SL(2,\mathbb{R})/Stab(X)}$.

unbounded

<u>Proof of the claim</u>:

Suppose not. Then $\exists$ sequence $t_i\to\infty$ s.t. $[G_{t_i}]$ stays in a cpt, so up to taking a subsequence we know that $\underbrace{[G_{t_i}]}_{=G_{t_i}\cdot Stab(X)}\longrightarrow\underbrace{[H]}_{=H\cdot Stab(X)}$ as $i\to\infty$.

So $\exists A_i\in Stab(X)$ such that $G_{t_i}\cdot A_i\longrightarrow H$ in $SL(2,\mathbb{R})$.

<u>Fact</u>: the fct $\mathcal{H}(k)\xrightarrow{sys}\mathbb{R}_{>0}$ is continuous.

$X\longmapsto$ length of a shortest saddle connection

So $\underbrace{sys(G_{t_i}A_i\cdot X)}_{=sys(G_{t_i}\cdot X),\ A_i\in Stab(X)}\longrightarrow sys(HX)<\infty$, a contradiction.

$\square$

<u>Lemma</u>

Let $\Gamma<SL(2,\mathbb{R})$ be a discrete subgroup s.t. $\mathbb{H}^2/\Gamma$ has finite volume. Then if $[G_t]\to\infty$ in $SL(2,\mathbb{R})/\Gamma$, $\exists s\neq 0$ s.t. $\begin{pmatrix}1&0\\s&1\end{pmatrix}\in\Gamma$.

The proof of the lemma is not hard, but it needs results about hyperbolic geometry.

Applying the lemma to $Stab(X)$ we get that $\exists \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in Stab(X)$, but we proved before that this implies that the vertical straight line flow is periodic.

<u>Rmk</u>: we can apply the lemma to $Stab(X)$ because X is a Veech surface, otherwise we would not have the hypothesis on the finiteness of the volume.

$\square$