# Riemann surfaces

Consider $S$ a top. surface (or., conn.). A <u>complex atlas</u> is a collection $\{(U_J, \varphi_J)\}_{J\in I}$ where

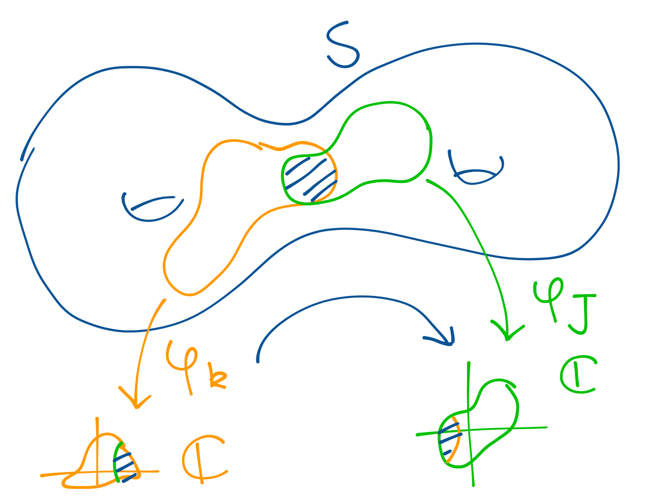
- $U_J \subseteq S$ open, $\bigcup_{J\in I} U_J = S$ [open cover of $S$]
- $\forall J$, $\varphi_J : U_J \longrightarrow \mathbb{C}$ is a homeomorphism onto an open set of $\mathbb{C}$ and $\forall J,k$: $U_J \cap U_k \neq \phi$, $\boxed{\varphi_J \circ \varphi_k^{-1}} : \varphi_k(U_J\cap U_k) \longrightarrow \varphi_J(U_J \cap U_k)$ is holomorphic.

$\varphi_k(U_J\cap U_k) \subseteq \mathbb{C}$, $\varphi_J(U_J \cap U_k) \subseteq \mathbb{C}$

local coord. ($\varphi_J$)

transition fct ($\varphi_J \circ \varphi_k^{-1}$)

<u>compatibility condition</u>




We'd want to say that a Riemann surface is $S$+ cpx atlas, but we have obvious things we want to identify:

<u>EX.</u> $\{(U_J, \varphi_J)\}_J$ atlas, pick an open set $V_{J_0} \subsetneq U_{J_0}$ and add $(V_{J_0}, \varphi_{J_0}|_{V_{J_0}})$ to the collection $\longrightarrow$ this should give the same object!

We say that two cpx atlases are <u>equivalent</u> if their union is still a cpx atlas.

<u>Cpx structure</u> = equivalence class of cpx atlases

<u>Riemann surface</u> = top. surf. $S$ + cpx structure on it

<u>EX.</u> (1) $\mathbb{C}$ with $\{(\mathbb{C}, id)\}$

(2) $S^2$ with ?

It's a surface using $\{(S^2\setminus N, p_N), (S^2\setminus S, p_S)\}$ $\longrightarrow$ interpret $p_N, p_S$ as maps to $\mathbb{C}$ instead of $\mathbb{R}^2$

To see if it's a complex atlas:

$$p_S \circ p_N^{-1}(z) = ?$$

Computations: $p_S \circ p_N^{-1}(x+iy) = \dfrac{x}{x^2+y^2} + i\,\dfrac{y}{x^2+y^2}$ $\longrightarrow$ <u>not</u> holomorphic (Cauchy-Riemann not satisfied)

Use instead $\overline{p_S}$ : $\overline{p_S}(p) := \boxed{\overline{p_S(p)}}$ $\leftarrow$ cpx conj.

Then $\overline{p_S} \circ p_N^{-1}(z) = \frac{1}{z} \rightarrow$ holomorphic?

$\downarrow$

not on $\mathbb{C}$, but yes on $p_N(S^2\setminus\{S,N\}) = \mathbb{C}\setminus\{0\}$

$\Longrightarrow$ the sphere has a cpx structure. We call this Riemann surface the <u>Riemann sphere</u>.

<u>Thm</u>: the sphere is rigid: $\exists!$ cpx structure on it.

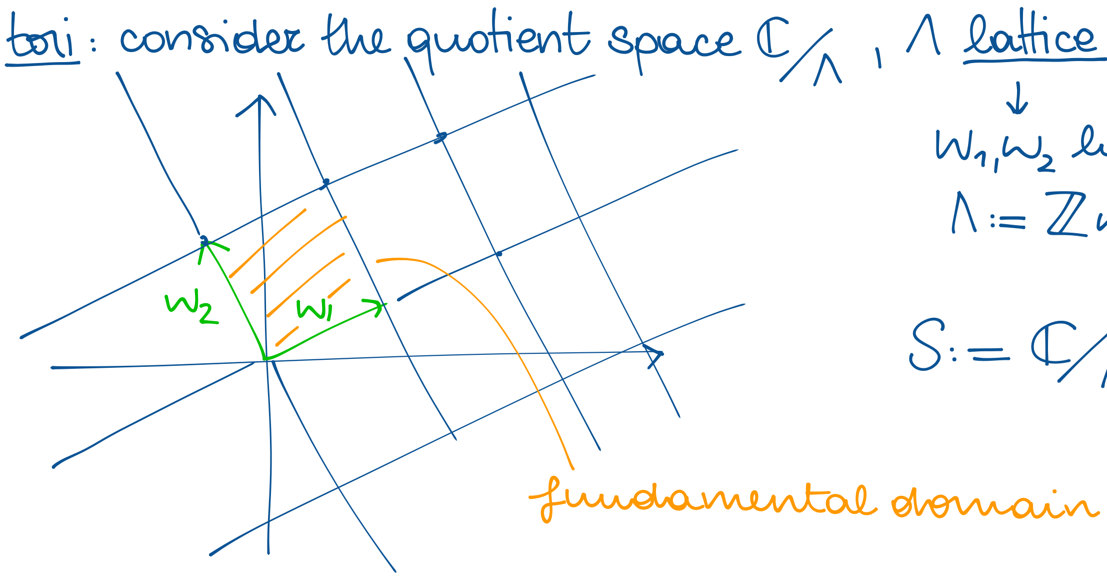
(3) <u>tori</u>: consider the quotient space $\mathbb{C}/\Lambda$, $\Lambda$ <u>lattice</u> in $\mathbb{R}^2$

$\downarrow$

$w_1, w_2$ lin. indep. vectors in $\mathbb{R}^2$

$\Lambda := \mathbb{Z}w_1 + \mathbb{Z}w_2 = \{n_1w_1 + n_2w_2 \mid n_i \in \mathbb{Z}\}$

$$S := \mathbb{C}/\Lambda$$



Complex structure: $\pi : \mathbb{C} \longrightarrow S$ is a local homeo

continuous and open (by def. of quotient top)

$d :=$ min distance btw two elements of $\Lambda \Rightarrow$

$\pi|_{B_\varepsilon(p)}$ is bijective $\forall \varepsilon < d/2$.

$\rightarrow$ charts: $(\underbrace{B_{d/2}(p)}_{U_p}, \underbrace{\pi|_{B_{d/2}(p)}^{-1}}_{\varphi_p})$

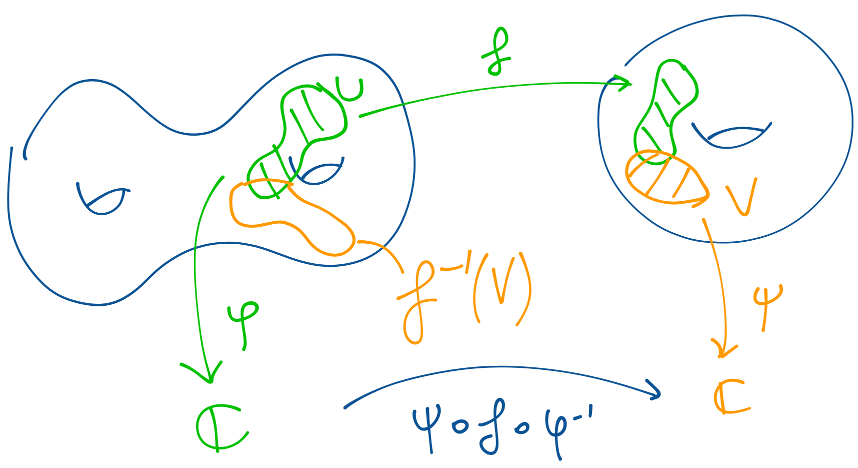
Supp. $U_p \cap U_q \neq \emptyset$ and consider

$$\varphi_p \circ \varphi_q^{-1} = \pi|_{U_p}^{-1} \circ \pi|_{U_q}$$

$$\Rightarrow \pi(\varphi_p \circ \varphi_q^{-1}(z)) = \pi(z) \quad \forall z \in \underbrace{U_p \cap U_q}_{\text{conn.}}$$

i.e. $\underbrace{\varphi_p \circ \varphi_q^{-1}(z) - z}_{\text{cont fct of } z} \in \Lambda$ discrete $\Bigg| \Rightarrow$ const: $\varphi_p \circ \varphi_q^{-1}(z) = z + \lambda_0$ holomorphic

$X, Y$ w/ a cpx structure $\rightarrow$ we can define $X \xrightarrow{f} Y$ holomorphic: $\forall (U, \varphi)$ chart of $X$, $\forall (V, \psi)$ chart of $Y$ s.t. $f(U) \cap V \neq \emptyset$, $\psi \circ f \circ \varphi^{-1}$ holomorphic where it is defined




$\rightarrow$ defined on $\varphi(U \cap f^{-1}(V)) \subseteq \mathbb{C}$

In particular, we can speak of holomorphic fcts from a Riemann surface to $\mathbb{C}$. Let's analyze some properties of these functions.

$f : X \longrightarrow \mathbb{C}$ holom, $p \in X$. Choose a chart centered at $p$: $(U, \varphi)$: $\varphi(p) = 0$

$\Rightarrow$ locally $f \circ \varphi^{-1}(z) = \sum_{i \geq k} a_i z^i$, where $k \geq 0$ and $a_k \neq 0$.

We say that $f$ has a zero of order $k$ at $p$ if $k \geq 1$.

Note that this is well defined: if $\psi$ is another chart centered at $p$, then locally

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ \underbrace{(\varphi \circ \psi^{-1})}_{\text{diffeo}}$$

$\Downarrow$

the order does not change.

Prop.

$f : X \rightarrow \mathbb{C}$ holomorphic, $p \in X$ zero of order $k$. Then $\exists$ loc. chart around $p$ wrt which $f$ is of the form $z \longmapsto z^k$.

Proof.

$(U, \varphi)$ chart centered at $p \Rightarrow f \circ \varphi^{-1}(z) = z^k g(z)$, where $g$ is holomorphic and $g(0) \neq 0$

$\exists$ nbhood of 0 in which we have a well-defined holom. k-th root $h(z)$

Set $\alpha(z) := z h(z)$.

Then $f \circ \varphi^{-1}(z) = (\alpha(z))^n$ (locally).

Define $\psi := \alpha \circ \varphi$ (loc. around 0)

We claim that this gives us the chart we want. We check first that it is a chart: $\psi \circ \varphi^{-1} = \alpha \rightarrow$ holomorphic and local diffeo because

$$\left.\frac{d\alpha}{dz}\right|_0 = \left.\left(g(z) + z\frac{dg}{dz}(z)\right)\right|_0 = g(0) \neq 0 \Rightarrow \alpha^{-1} \text{ holomorphic too.}$$

Furthermore in this chart we have:

$$f\circ\varphi^{-1}(z) = (f\circ\varphi^{-1})(\alpha^{-1}(z)) = (\alpha(\alpha^{-1}(z)))^k = z^k.$$

$\square$

Prop.

$X$ cpt Riemann surface, $f: X \to \mathbb{C}$ holomorphic $\Rightarrow$ $f$ constant.

We will use:

Thm (Open mapping thm)

$U \subseteq \mathbb{C}$ domain, $g: U \to \mathbb{C}$ holomorphic and not constant. Then $g$ is open.

Proof of the prop.

Suppose $f$ is not constant. Using the open mapping theorem we can show that $f$ is open, so $f(X)$ is open. Furthermore, since $X$ is cpt and $f$ continuous, $f(X)$ is cpt in $\mathbb{C}$, hence closed. But then $f(X) \neq \emptyset$ is a connected component of $\mathbb{C}$ and $\mathbb{C}$ is connected, hence $f(X) = \mathbb{C}$. But $f(X)$ is cpt and $\mathbb{C}$ isn't, a contradiction.

$\square$

Abelian differentials

$X$ Riemann surface.

An Abelian differential is the datum, $\forall$ chart $(U, z)$, of a holomorphic function $f: z(U) \to \mathbb{C}$ w/ the compatibility condition:

$$\begin{matrix} (U, z) \rightsquigarrow f(z) \\ (V, w) \rightsquigarrow g(w) \end{matrix} \implies f(z)\frac{dz}{dw} = g(w), \text{ where we see } z = z(w) \text{ via the transition function}$$

Alternatively: an Abelian differential is a holomorphic 1-form on $X$

Informally: $\boxed{\text{locally}}$ "$f(z)dz$", $f$ holomorphic

in a chart

EX. torus $\mathbb{C}/_\Lambda$, $\Lambda = \mathbb{Z} + i\mathbb{Z}$

Charts: $\left(U_p, \pi|_{B_\varepsilon(q)}^{-1}\right)$ where $p \in \mathbb{C}/_\Lambda$, $q \in \pi^{-1}(p)$, $\varepsilon$ smaller than half the distance between two lattice pts, $U_p = \pi(B_\varepsilon(q))$

In each such chart: $1$

Transition fcts: $z \mapsto \underset{\substack{\parallel \\ w}}{z+c} \implies \frac{dz}{dw} = 1 \rightsquigarrow$ the compatibility condition are satisfied

EX sphere

$$\begin{matrix} S^2 \supseteq S^2 \setminus S & \longrightarrow & \mathbb{C} \text{ coord } w \\ \cup\!| & & \\ S^2 \setminus N & & \\ \downarrow & & \\ \mathbb{C} & & \\ \text{coord } z & & \end{matrix}$$

Coordinate change: $z = \frac{1}{w}$

Choose $f_N(z) = \sum_{i\geq 0} a_i z^i \implies f_S(w)$ must be

$$f_S(w) = f_N\left(\tfrac{1}{w}\right)\frac{dz}{dw} = \left(\sum_{i\geq 0} a_i w^{-i}\right)\left(\frac{-1}{w^2}\right) =$$

$$= -\sum_{i\geq 0} a_i w^{-i-2} \to \text{it has a pole at } w=0 \text{ unless } a_i = 0\ \forall i$$

$\Rightarrow \nexists$ nonzero $\omega$

Like for holomorphic functions we can speak about zeroes of an Abelian diff. and their order:

$p \in X$ is a zero of $\omega$ of order $k$ if $\exists$ chart $(U, \varphi)$ centered at $p$ such that the fct associated to $\omega$ in this chart is of the form $z^k g(z)$, where $g$ is a holomorphic fct with $g(0) \neq 0$.

This is well defined: if $(V,\psi)$ is another chart centered at $p$ and $f(w)$ is the fct associated to $\omega$ in this chart, then

$$g(w) = f(z(w))\boxed{\frac{dw}{dz}} \to \neq 0 \text{ at } 0$$

sends 0 to 0 and has non-zero der. at 0

$\Rightarrow$ same order of zero

Given a zero $p$ of $\omega$, we denote by $k_p$ its order of zero. Then we have

Thm (Riemann-Roch)

$X$ cpt Riemann surface of genus $g \geq 1$. Then for any non-zero Abelian differential $\omega$ we have

$$\sum_{p \text{ zero of } \omega} k_p = 2g-2.$$

In words: $\omega$ has $2g-2$ zeroes, counted with multiplicity.

Note: while it makes sense to say that $\omega$ vanishes (or not) at a point $p$, the value of $\omega$ at a point is not defined. Indeed, given a chart $(U,\varphi)$ in which $\omega$ corresponds to $f(z)$, for any biholomorphic fct $g: \varphi(U) \to V \subseteq \mathbb{C}$ we have a new chart $(U, g\circ\varphi)$. In this chart, $\omega$ is given by

$$h(w) = f(z(w))\frac{dz}{dw} = f(g^{-1}(w))\frac{dg^{-1}(w)}{dw}$$

hence unless $\frac{dg^{-1}}{dw}(g(0)) = 1$, $h(g(0)) \neq f(0)$.

EX: for the torus, pick $p \in \mathbb{C}/\Lambda$ and a chart $(U_p, \varphi_p)$ defined before. Then $\varphi_p(U_p) = B_\varepsilon(q)$ for some $q \in \mathbb{C}$. Consider $g: B_\varepsilon(q) \to \mathbb{C}$

$$z \longmapsto (z-q+1)^2 = w$$

This is a biholomorphism, so we get a new chart $(U_p, g\circ\varphi_p)$. In this chart the $\omega_0$ which is 1 in all "canonical" charts is given by

$$1 \cdot \frac{1}{\frac{dg}{dz}(g^{-1}(w))} = \frac{1}{2\sqrt{w}}$$

and at $z = q$ (corresponding to $p$) it is not 1.

Note that this is also not constant!

Rmk: $\omega_1, \omega_2$ nonzero Abelian diff. on $X$ w/ same zeroes of the same order. Then $\exists \lambda \in \mathbb{C}\setminus\{0\}$ s.t. $\omega_1 = \lambda\omega_2$.

Proof. For any chart $(U,\varphi)$, $\omega_j$ correspond to $f_j$ holom. on $U$. Since $f_1$ & $f_2$ have the same zeroes, $f_2/f_1$ is also holomorphic. We claim that these fcts glue together to give a holomorphic fct on $X$. Indeed, both fcts change by the same factor when changing the chart (the derivative of the transition fct) $\Rightarrow$ globally defined map. But then this must be constant, and $\neq 0$ since $\omega_1 \neq 0$. □

As a consequence, on the torus we have a 1-(complex) parameter family of Ab. differentials, all constant multiples of the one we constructed before.

Next we want to show that actually we can always find charts so that $\omega$ is locally constant around points that are not zeroes. For this, we will define the integral of an Abelian differential along a curve.

Let $\gamma:[0,1]\to X$ be a smooth curve and $(U_1,\varphi_1),\dots,(U_k,\varphi_k)$ charts such that $\exists$ $0=t_0<t_1<\dots<t_{k-1}<t_k=1$ for which $\gamma([t_{J-1},t_J])\subseteq U_J$ $\forall J\in\{1,\dots,k\}$. Define

$$\int_\gamma \omega := \sum_{J=1}^{k} \int_{\varphi_J\circ\gamma|_{[t_{J-1},t_J]}} f_J(z)\,dz$$

where $f_J$ is the holomorphic fct representing $\omega$ in the chart $(U_J,\varphi_J)$.

Rmk: $\exists$ a finite collection of charts with the property above because $\gamma([0,1])$ is cpt

Lemma

Suppose $\gamma$ is a smooth curve and $(U,\varphi),(V,\psi)$ charts such that $\operatorname{Im}\gamma\subseteq U\cap V$ and let $f,g$ be the fcts representing $\omega$ in $(U,\varphi),(V,\psi)$. Then

$$\int_{\varphi\circ\gamma} f(z)\,dz = \int_{\psi\circ\gamma} g(w)\,dw.$$

Proof.

$$\int_{\varphi\circ\gamma} f(z)\,dz \underset{\substack{\uparrow\\ \text{change of}\\ \text{coordinates}}}{=} \int_{\psi\circ\gamma} f(w(z))\,\frac{dz(w)}{dw}\,dw \underset{\substack{\uparrow\\ \text{compatibility}\\ \text{conditions for } \omega}}{=} \int_{\psi\circ\gamma} g(w)\,dw.$$

$\square$

Let now $\gamma:[0,1]\to X$ be a smooth curve and $(U_1,\varphi_1),\dots,(U_k,\varphi_k)$ charts s.t. there exist $0=t_0<t_1<\dots<t_k=1$ for which $\gamma([t_{J-1},t_J])\subseteq U_J$. Let $f_k$ be the function representing $\omega$ in $(U_k,\varphi_k)$. We define

$$\int_\gamma \omega := \sum_{i=1}^{k} \int_{\varphi_i\circ\gamma|_{[t_{i-1},t_i]}} f_k(z)\,dz.$$

Rmk: such collection of charts exist because $\gamma([0,1])$ is compact, so we can always find a finite subcover.

Lemma

The definition is independent of the choice of charts.

Proof. Suppose $(V_1,\psi_1),\dots(V_m,\psi_m)$ is another set of charts with functions $g_1,\dots,g_m$ representing $\omega$ and points $0=s_0<\dots<s_m=1$ with $\gamma([s_{J-1},s_J])\subseteq V_J$. We can refine the partition of $[0,1]$ to $0<u_0<u_1<\dots<u_\ell=1$ so that all $t_J$ and $s_J$ appear and $\forall J$ $\gamma([u_{J-1},u_J])$ is contained in $U_a\cap V_b$ for some $a$ and $b$. By the additivity of integrals, we just need to check that

$$\int_{\varphi_a\circ\gamma|_{[u_{J-1},u_J]}} f_a(z)\,dz = \int_{\psi_b\circ\gamma|_{[u_{J-1},u_J]}} g_b(w)\,dw$$

which is the content of the lemma before.

$\square$

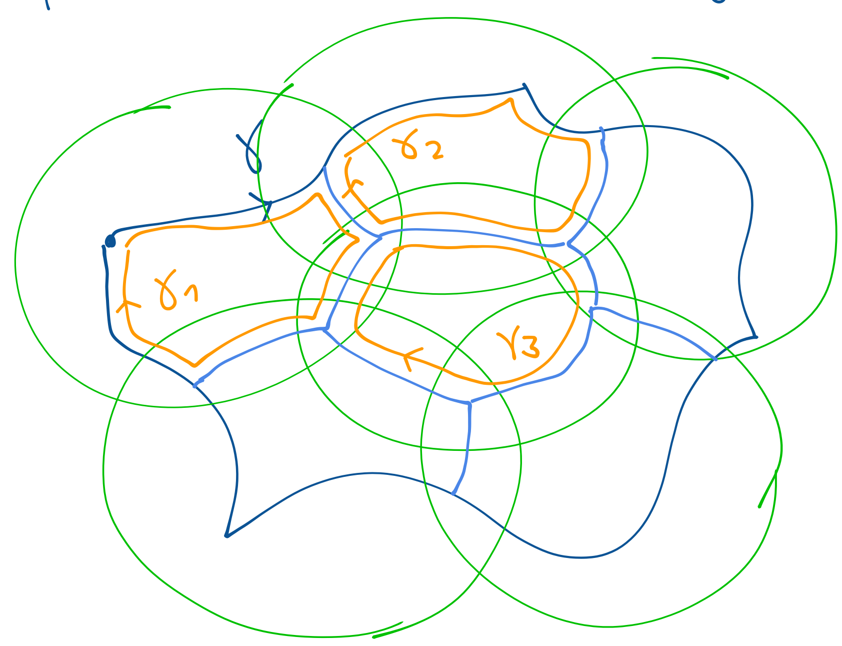
If $\gamma$ is piecewise smooth, we can define $\int_\gamma \omega$ as the sum of the integrals on the smooth pieces.

Lemma

If $\gamma$ is piecewise smooth and bounds a topological disk (i.e. one connected component of $X\setminus\gamma$ is homeomorphic to a disk), then $\int_\gamma\omega=0$.

Proof.

Let $\{(U_J,\varphi_J)\}_J$ be a finite collection of charts covering the closed disk bounded by $\gamma$ (i.e. the topological disk and $\gamma([0,1])$). Subdivide the disk into smaller topological disks bounded by pieces of $\gamma$ and/or piecewise smooth curves so that each piece is contained in a single chart:




Let $\gamma_{J_i}$ be the boundaries of the disks in $U_J$, with orientation induced by the orientation of $\gamma$ when a piece of $\gamma$ is part of $\gamma_{J_i}$ and so that each piece common to two disks has opposite orientations. Then by basic properties of the integral

$$\int_\gamma \omega = \sum_J \sum_i \int_{\gamma_{J_i}} \omega .$$

But since $\omega$ is given by holomorphic functions in local charts, the Residue Theorem tells us that each summand is zero and thus $\int_\gamma \omega = 0$.

$\square$

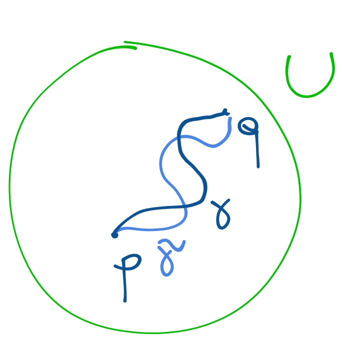
Using this we show:

Prop.
$X$ Riemann surface, $\omega \not\equiv 0$ Abelian differential. Then $\forall p \in X$ $\exists$ chart centered at $p$ in which $\omega$ is given by the function $1$, if $p$ is not a zero of $\omega$, or by $z^k$ if $p$ is a zero of $\omega$ of order $k$. Furthermore, the transition fct between two such charts around regular points is a translation.

Proof.
Let $(U,\varphi)$ be a chart centered at $p$ and suppose (up to restricting $U$) that $U$ is simply connected and does not contain any zero of $\omega$ (except $p$, if it is one).
Define the map $U \longrightarrow \mathbb{C}$

$$q \longmapsto \int_\gamma \omega \quad \text{for any piecewise smooth path } \gamma \text{ from } p \text{ to } q.$$

This is well defined: if $\tilde{\gamma}$ is another such path, $\gamma$ and $\tilde{\gamma}$ intersect finitely many times and the integral of the two paths between consecutive intersection points are the same by the previous lemma:




We denote $\int_\gamma \omega$ by $\int_p^q \omega$.

Let $g(w)$ be the function representing $\omega$ in $(U,\varphi)$, where $g$ is holomorphic vanishing of order $k \geq 0$ (where if $k=0$ we mean that $g$ does not vanish) at $0$. Then

$$\int_p^q \omega = \int_0^{\boxed{w=w(q)}} g(t)\,dt$$

— local coordinate for $\varphi$

is zero at $w=0$ and its derivative at $0$ is $g(0)$, which vanishes of order $k$. So the integral vanishes of order $(k+1)$ at $0$ and hence (similarly to what we saw before) it has a holomorphic $(k+1)$-st root. Set

$$z(w) := \left( (k+1) \int_0^w g(t)\,dt \right)^{1/k+1}$$

Claim: this is the chart we want.

(a) it is a chart because the fct $z = z(w)$ is holomorphic and

$$\frac{dz}{dw} = \frac{1}{k+1} \frac{1}{\boxed{\underbrace{\left((k+1)\int_0^w g(t)\,dt\right)}_{\text{vanishes of order } k+1 \text{ at } 0}^{k/k+1}}_{\text{vanishes of order } k \text{ at } 0}} (k+1) \underbrace{g(w)}_{\text{Vanishes of order } k \text{ at } 0}$$

$\Longrightarrow \frac{dz}{dw}(0) \neq 0 \Longrightarrow z(w)$ is biholomorphic.

(b) if $f$ is the fct associated to $\omega$ in this chart, we must have

$$f(z) \frac{dz}{dw} = g(w) \qquad \Longrightarrow f(z) = z^k$$

$$\| $$

$$f(z) \frac{g(w)}{z^k}$$

Transition maps: $p_1, p_2$ s.t. $\omega(p_i) \neq 0$, charts $z_i(q) = \int_{p_i}^q \omega$. Assume $U_1 \cap U_2$ simply connected.

Then: $z_1(q) = \int_{p_1}^q \omega = \int_{p_1}^{p_2} \omega + \int_{p_2}^q \omega = z_2(q) + \underbrace{\int_{p_1}^{p_2} \omega}_{\substack{\text{constant} \\ \text{(indep. of } q)}}$

$\square$