Parameter spaces of translation surfaces

We want to define two parameter spaces of translation surfaces, one of which is a cover of the other. We will use the (X,ω) definition first and start defining parameter spaces of Riemann surfaces.

Fix a top. cpt surface S of genus $g \geq 1$.
The <u>Teichmüller space</u> of S is the space

$$\mathcal{T}_g := \left\{ (X,f) \,\middle|\, X \text{ Riemann surface}, f: S \to X \overset{\text{or. pres.}}{\text{homeomorphism}} \right\} \Big/ \sim$$

where $(X,f) \sim (Y,h)$ if $\exists$ biholomorphism $\varphi: X \to Y$ s.t. $\varphi \circ f$ is homotopic to h, i.e. the diagram

$$\begin{tikzcd} & X \arrow[d, "\varphi"] \\ S \arrow[ur, "f"] \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes up to homotopy. The map f is called <u>marking</u>.

The <u>moduli space</u> of S is

$$\mathcal{M}_g = \{ X \text{ Riemann surface of genus } g \} \Big/ \text{biholomorphism}$$

Note that there is a natural projection $\mathcal{T}_g \longrightarrow \mathcal{M}_g$ – forgetting the marking.
$$[(X,f)] \longmapsto [X]$$

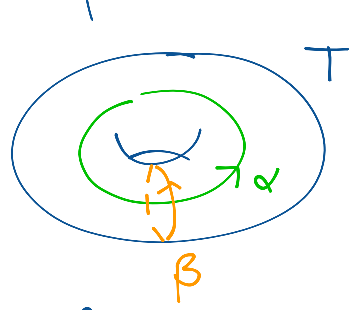
How are the two spaces different? If X is a RS without non-trivial self-biholomorphisms and $\varphi: S \longrightarrow S$ is a homeomorphism not homotopic to the identity, $[(X,f)]$ and $[(X, f\circ\varphi)]$ are different in $\mathcal{T}_g$:

$$\begin{tikzcd} & X \arrow[d, "\mathrm{id}"] \\ S \arrow[ur, "f"] \arrow[r, "f\circ\varphi"'] & X \end{tikzcd}$$

id (only bihol.) $\Rightarrow$ need $f \sim f \circ \varphi$, i.e. $\varphi \sim \mathrm{id}$ ↯

<u>EX</u>: $\boxed{g=1}$ One can show (and we will assume) that any cpx torus is bihol. to $\mathbb{C}/\Lambda$, for Λ a lattice.

Let T be a top. torus and choose an equator α and a meridian β of T:



A marking $f: T \to \mathbb{C}/\Lambda$ gives us then, looking at the images of α and β, a basis $\{v_1, v_2\}$ of Λ over $\mathbb{Z}$, where v_1 and v_2 are positively oriented.
Conversely, an oriented basis of Λ gives us a homeomorphism with T (defined up to homotopy): send α, β to the curves given by the basis vectors and complete with a homeomorphism of the two leftover disks respecting the maps on the curves.

<u>Prop.</u>
$\mathcal{T}_1 \simeq \{ \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \}$.

<u>Proof</u>. Let $[(X,f)] \in \mathcal{T}_1 \Rightarrow X = \mathbb{C}/\Lambda$, $\Lambda = \langle v_1, v_2 \rangle$ or. basis (as above). The map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z/v_1$, induces a biholomorphism $\varphi: X \longrightarrow \mathbb{C}/\langle 1, v_2/v_1 \rangle$

and $[(X,f)] = [(\mathbb{C}/\langle 1, \frac{v_2}{v_1}\rangle, \varphi\circ f)]$. Since v_1, v_2 is an oriented basis, $\operatorname{Im}\frac{v_2}{v_1} > 0$.
So each $[(X,f)] \in \mathcal{T}_1$ has a representative of the form $X_\tau = \mathbb{C}/\langle 1,\tau\rangle$, where $\operatorname{Im}\tau > 0$. We just need to show that if $\tau \neq \tau'$, X_τ and $X_{\tau'}$ define different elements in $\mathcal{T}_1$. Supp. $\varphi : X_\tau \longrightarrow X_{\tau'}$ biholomorphism. Then φ lifts to $\tilde{\varphi} : \mathbb{C} \longrightarrow \mathbb{C}$ and $\tilde{\varphi}$ is a biholomorphism. Hence $\tilde{\varphi}(z) = \alpha z + \beta$ (cpx analysis fact) and we have

$$\begin{tikzcd} \tilde{\varphi}: \mathbb{C} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "/\!/"] & \mathbb{C} \arrow[d, "\pi'"] \\ X_\tau \arrow[r, "\varphi"'] & X_{\tau'} \end{tikzcd}$$

So $\tilde{\varphi}(0) \in \langle 1, \tau'\rangle$, hence wlog. $\tilde{\varphi}(0) = 0$. Furthermore if φ respects the markings (which correspond to bases), $\tilde{\varphi}$ must send 1 to 1 and τ to τ'. Thus $1 = \tilde{\varphi}(1) = \alpha \Rightarrow \tau' = \tau$.

$\square$

The moduli space is a quotient of $\mathcal{T}_1$:

<u>Prop.</u>
$\mathcal{M}_1 \simeq \{\tau \mid \operatorname{Im}\tau > 0\}/SL(2,\mathbb{Z})$, where $\underset{\underset{SL(2,\mathbb{Z})}{\cap}}{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

<u>Proof</u>: the map $\underset{\underset{\mathbb{C}}{\cap}}{z} \longmapsto \underset{\underset{\mathbb{C}}{\cap}}{(c\tau+d)}z$ induces a biholomorphism

$$\mathbb{C}/\langle 1, \tfrac{a\tau+b}{c\tau+d}\rangle \longrightarrow \mathbb{C}/\langle a\tau+b, c\tau+d\rangle$$

but since $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$, $\langle a\tau+b, c\tau+d\rangle = \langle 1, \tau\rangle$, so $X_\tau \simeq X_{\frac{a\tau+b}{c\tau+d}}$.

Conversely, suppose $\varphi : X_\tau \longrightarrow X_{\tau'}$ is a biholomorphism. Then as before we can lift this to $\tilde{\varphi} : \mathbb{C} \to \mathbb{C}$ of the form $\tilde{\varphi}(z) = \alpha$. Since φ is a biholom, $\underset{\underset{\langle \alpha, \alpha\tau\rangle}{\|}}{\tilde{\varphi}(\langle 1, \tau\rangle)} = \langle 1, \tau'\rangle$, i.e. $\exists\, a,b,c,d \in \mathbb{Z}$ s.t. $\begin{cases} \alpha\tau = a\tau' + b \\ \alpha = c\tau' + d \end{cases}$

$$\exists\, a', b', c', d' \in \mathbb{Z} : \begin{cases} \tau' = a'(\alpha\tau) + b'\alpha \\ 1 = c'(\alpha\tau) + d'\alpha \end{cases}$$

$$\Rightarrow \begin{cases} \tau' = a'(a\tau'+b) + b'(c\tau'+d) \\ 1 = c'(a\tau'+b) + d'(c\tau'+d) \end{cases}$$

1 and τ' lin. indep $\Rightarrow \begin{cases} a'a + b'c = 1 \\ a'b + b'd = 0 \\ c'a + d'b = 0 \\ c'b + d'd = 1 \end{cases}$ i.e. $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Id}$

$\Rightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$ (since it has integral coefficients)

Furthermore $\tau = \frac{a\tau'+b}{\alpha} = \frac{a\tau'+b}{c\tau'+d}$ and

$$\underset{\overset{\vee}{0}}{\operatorname{Im}\tau} = \frac{1}{|c\tau'+d|^2} \operatorname{Im}\big((a\tau'+b)(c\overline{\tau'}+d)\big) = \underset{>0}{\frac{1}{|\quad|^2}}(ad-bc)\underset{\vee 0}{\operatorname{Im}\tau'}$$

$\Rightarrow ad - bc > 0$, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

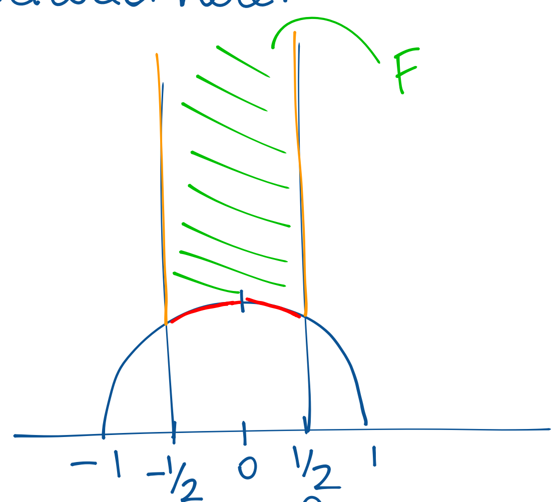
$\square$

<u>Rmk</u>: $A\cdot\tau = (-A)\cdot\tau \Rightarrow$ we have an action of $PSL(2,\mathbb{Z})$ on $\mathcal{T}_1$, with quotient $\mathcal{M}_1$.

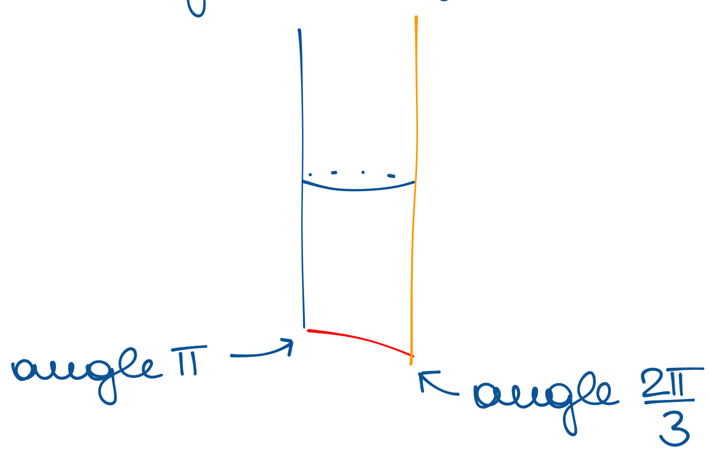
A fundamental domain for the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H} := \{\operatorname{Im}\tau > 0\}$ is the set $F := \{\tau \in \mathbb{H} \mid |\operatorname{Re}(\tau)| \le 1/2,\ |\tau| \ge 1\}$. By this we mean that

- $\forall \tau \in \mathbb{H}\ \exists A \in PSL(2,\mathbb{Z})$ s.t. $A \cdot \tau \in F$
- $\forall \tau \neq \tau' \in \underbrace{\operatorname{Int}(F)}_{\text{interior of } F},\ \nexists A \in PSL(2,\mathbb{Z}) \setminus \{Id\}$ s.t. $A \cdot \tau = \tau'$.

So all identifications left are between points on ∂F. One can check that they are the ones pictured here:



So $\mathcal{M}_1$ is the singular surface obtained as $F/PSL(2,\mathbb{Z})$:



Let $MCG_g := \underbrace{\text{Homeo}^+(S)}_{\substack{\text{orientation}\\ \text{preserving}\\ \text{homeos}}} \Big/ \underbrace{\text{Homeo}_0(S)}_{\text{homeos homotopic to id}}$

It is a group, called <u>mapping class group</u>.

Note that MCG_g acts on $\mathcal{T}_g$ by precomposition of the marking: $[\varphi] \in MCG_g$, $[(X,f)] \in \mathcal{T}_g \Rightarrow [\varphi] \cdot [(X,f)] = [(X, f\circ\varphi)]$.

This is well defined: supp. $[\psi] = [\varphi] \in MCG_g$ and $[(X,f)] \in \mathcal{T}_g$. Then $id: X \to X$ is a biholomorphism and $id \circ f \circ \varphi$ is homotopic to $f \circ \psi$, since $\varphi \sim \psi$, so $[(X, f\circ\varphi)] = [(X, f\circ\psi)]$.

If $[(X,f)] = [(Y,h)] \in \mathcal{T}_g$ and $[\varphi] \in MCG_g$, then $\exists F: X \to Y$ biholom. s.t. $F \circ f \sim h \Rightarrow F \circ f \circ \varphi \sim h \circ \varphi$, so

$$[(X, f\circ\varphi)] = [(Y, h\circ\varphi)].$$

<u>Lemma</u>

The projection map $\mathcal{T}_g \to \mathcal{M}_g$ descends to a bijection $\mathcal{T}_g / MCG_g \to \mathcal{M}_g$.

<u>Proof</u>: the map descends because the MCG_g action only changes the marking. Supp. $[(X,f)]$ and $[(Y,h)] \in \mathcal{T}_g$ induce the same elt in $\mathcal{M}_g$. Then $\exists F: X \to Y$ biholomorphism and we set $\varphi := h^{-1} \circ F \circ f \in \text{Homeo}^+(S)$. Then $[\varphi] \cdot [(Y,h)] = [(Y, F\circ f)]$ and since the diagram

$$\begin{tikzcd} & X \arrow[d, "F"] \\ S \arrow[ur, "f"] \arrow[r, "F\circ f"'] & Y \end{tikzcd}$$

commutes, $[(Y, F\circ f)] = [(X,f)]$, i.e. $[\varphi] \cdot [(Y,h)] = [(X,f)]$. $\square$

In the case $g=1$, one can show that $MCG_1 \simeq PSL(2,\mathbb{Z})$, which is coherent with what we proved before.

One way to put a topology on $\mathcal{T}_g$ is to define a metric on it. For this we need to speak about quasiconformal maps.

A map $f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is <u>K-quasiconformal</u> (K-qc, $K \geq 1$) if $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are locally in L^2 and $\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \underbrace{\boxed{\frac{K-1}{K+1}}}_{<1} \left|\frac{\partial f}{\partial z}\right|$ almost everywhere.

<u>$K(f)$</u> := smallest K s.t. f is K-qc = <u>qc constant</u> of f.

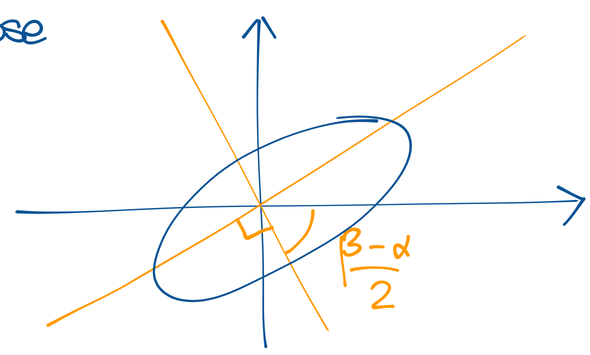
What does $K(f)$ measure? Suppose f is $\mathcal{C}^1(U)$ and K-qc. Then

$$Df(z_0) \text{ is } \mathbb{R}\text{-linear} \rightsquigarrow Df(z_0)(u) = \underbrace{\boxed{\frac{\partial f}{\partial z}(z_0)}}_{= a} u + \underbrace{\boxed{\frac{\partial f}{\partial \bar{z}}(z_0)}}_{= b} \bar{u}.$$

Assume $|a| > |b|$.

Look at the preimage of $|u|=1$: setting $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$

$|Df(z_0)(u)| = 1 \rightsquigarrow$ ellipse



minor axis of length $\frac{2}{|a|+|b|}$

major axis of length $\frac{2}{|a|-|b|}$

$$\Rightarrow \text{ratios of the axes: } \frac{|a|+|b|}{|a|-|b|} = \frac{\left|\frac{\partial f}{\partial z}(z_0)\right| + \left|\frac{\partial f}{\partial \bar{z}}(z_0)\right|}{\left|\frac{\partial f}{\partial z}(z_0)\right| - \left|\frac{\partial f}{\partial \bar{z}}(z_0)\right|} \leq K \quad (\text{since } f \text{ } K\text{-qc}).$$

So $K(f)$ bounds the "local distortion" of f at each point.

If $K(f) = 1$, f is called <u>conformal</u>.

A conformal map is one that preserves angles.

A $\mathcal{C}^1$-map $f: X \to Y$, for X, Y Riemann surfaces, is <u>K-qc</u> if it is K-qc in all charts.

We define, for $[(X,f)], [(Y,h)] \in \mathcal{T}_g$:

$$d_T([(X,f)],[(Y,h)]) := \inf\{\log(K(\varphi)) \mid \varphi: X \to Y \text{ qc } \mathcal{C}^1 \text{homeo and } h \text{ homotopic to } \varphi \circ f\}.$$

<u>Thm</u>

d_T is a distance. With the topology induced by d_T, $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ if $g \geq 2$.

We give $\mathcal{M}_g$ the quotient topology. MCG_g acts "<u>nicely</u>" on $\mathcal{T}_g$ and $\mathcal{M}_g$ is an orbifold (i.e. it has "cone points") of dimension $6g-6$.

properly discontinuously:
$\forall K \subseteq \mathcal{T}_g$ cpt, the set
$\{[\varphi] \in MCG_g \mid [\varphi]K \cap K = \emptyset\}$
is finite

d_T is the <u>Teichmüller distance</u>.

EX $g=1 \Rightarrow (\mathcal{T}_1, d_T) \simeq \boxed{\mathbb{H}^2} \simeq \left(\{\operatorname{Im}\tau>0\}, \boxed{\frac{|d\tau|}{\operatorname{Im}\tau}}\right)$

hyp. plane

metric of const. curvature -1

$\mathcal{M}_1$ is a hyperbolic orbifold, with a cone pt of angle π, one of angle $\frac{2\pi}{3}$ and a cusp.

top. a cylinder, but it has finite area

The metric is complete: the cusp is "infinitely long".

Let us now add the Abelian differential. Note that we can already separate translation surfaces of genus g depending on the number and order of zeroes. So fix $k=(k_1,\dots,k_n)$ collection of positive integers such that $\sum_{i=1}^n k_i = 2g-2$ and S a genus g surface with n marked pts $\{p_1,\dots,p_n\}=\Sigma$. For ω an Abelian differential, let $Z(\omega) := \{\text{zeroes of } \omega\}$.

$$\mathcal{TH}(k) := \left\{ [(X,\omega,f)] \;\middle|\; (X,\omega) \text{ translation surface}, \omega \text{ with } n \text{ zeroes of order } k_1,\dots,k_n,\ f:(S,\Sigma)\to(X,Z(\omega)) \text{ or. pres. homeo} \right\}$$

where $(X,\omega,f)\sim(Y,\eta,h)$ if $\exists F: X \to$ biholomorphism s.t. $h \sim F\circ f$ rel Σ and ω is the pull-back of η via F $(\omega = F^*\eta)$

For (U,φ) chart of X and (V,ψ) chart of Y small enough s.t. $F(U)\subseteq V$, $F^*\eta$ is given by the function $h\circ\psi\circ F\circ\varphi^{-1}(z)\,\frac{d(\psi\circ F\circ\varphi^{-1})(z)}{dz}$ on $\varphi(U)$, where h is the function defining η on in the chart (V,ψ)

Note: f is a map of pairs, i.e. it also sends Σ to $Z(\omega)$ (bijectively since it is a homeomorphism)

The Teichmüller space of translation surfaces of genus g is the union of these spaces:

$$\mathcal{TH}_g := \bigcup_{\substack{k \text{ s.t.} \\ \sum k_i = 2g-2}} \mathcal{TH}(k).$$

We also define

$$\mathcal{H}(k) := \left\{ (X,\omega) \;\middle|\; (X,\omega) \text{ transl. surf}, \omega \text{ has } n \text{ zeroes of order } k_1,\dots,k_n \right\}/\sim$$

where $(X,\omega)\sim(Y,\eta)$ if $\exists F: X\to Y$ biholomorphism: $\omega = F^*\eta$.

The moduli space of translation surfaces of genus g is

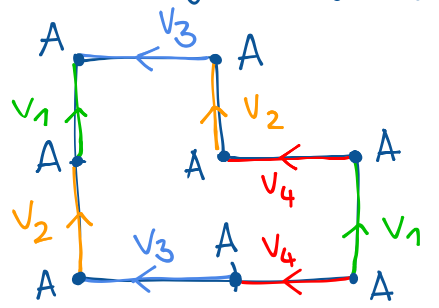
$$\mathcal{H}_g := \bigcup_{\substack{k \text{ s.t.} \\ \sum k_i = 2g-2}} \mathcal{H}(k).$$

By Riemann-Roch, the condition $2g-2 = \sum_{i=1} k_i$ is necessary to have $\mathcal{TH}(k)\neq\emptyset$ (resp. $\mathcal{H}(k)\neq\emptyset$). A priori it is not obvious that this is sufficient, but it can be shown to be.

Note also that there are obvious projections $\begin{matrix}\mathcal{TH}(k) \\ \downarrow \\ \mathcal{H}(k)\end{matrix}$, $\begin{matrix}\mathcal{TH}_g \\ \downarrow \\ \mathcal{H}_g\end{matrix}$

and as for the case of Riemann surfaces, an action of MCG_g on $\mathcal{TH}(k)$, $\mathcal{TH}_g$ with quotients $\mathcal{H}(k)$, $\mathcal{H}_g$. If instead of the marking we forget the diff we get maps $\mathcal{TH}_g \to \mathcal{T}_g$, $\mathcal{H}_g \to \mathcal{M}_g$, $\mathcal{TH}(k)\to\mathcal{T}_g$, $\mathcal{H}(k) \longrightarrow \mathcal{H}_g$.

We defined the parameter spaces in terms of Riemann surfaces and Abelian differentials. Similarly, one could define them using singular flat structures or gluings of polygons. In both cases, the Teichmüller space is the space of marked structure and the moduli space is the space of unmarked ones. An intuitive description of which topology the spaces we defined are endowed with can be given in terms of gluings of polygons. Supp. $X \in \mathcal{T}_g$ is given by



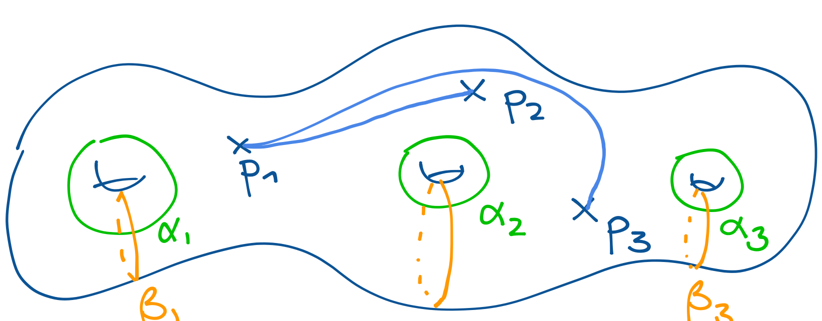
Then a nbhood of X is given by surfaces obtained by "wiggling a bit" the polygon: change the v_i's to slightly different vectors and consider the surfaces we obtain like this.

The problem is that it is unclear how many parameters we have – what is the (local) dimension of the space: we could have a representation of the same surface with many more polygons/edges.

The solution is to look again at Riemann surfaces and Abelian differentials and show that local coordinates for $\mathcal{TH}(k)$, $k=(k_1,\dots,k_n)$, are given by integrating the differential on a well-chosen set of curves. Precisely, on S with n marked pts $p_1,\dots,p_n$, we choose curves $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ and $\gamma_1,\dots,\gamma_{n-1}$ as follows: $\alpha_1,\dots,\beta_g$ are nonsep. simple closed curves. α_i intersects only β_i once.

$\downarrow$ $S \setminus \alpha_i$ or $S\setminus\beta_i$ connected

γ_i is a simple path connecting p_1 to p_i, disjoint (except at p_1) from any other curve. Concretely:



Prop.

The map $[(X,\omega,f)] \longmapsto \left(\int_{f\circ\alpha_1}\omega,\dots,\int_{f\circ\alpha_g}\omega,\int_{f\circ\beta_1}\omega,\dots,\int_{f\circ\beta_g}\omega,\int_{f\circ\gamma_1}\omega,\dots,\int_{f\circ\gamma_{n-1}}\omega\right)\in\mathbb{C}^{2g+n-1}$ gives local coordinates on $\mathcal{TH}(k)$ (called period coordinates).

The idea is that the curves $\alpha_1,\dots,\gamma_{n-1}$ form a basis for $H_1(S,\{p_1,\dots,p_n\};\mathbb{Z})$ and integrating ω is an operation that is well-defined on relative hom. classes. Since the curves form a basis, the coordinates determine the lengths of the sides of triangles which can be constructed from (X,ω) as we saw. So one can reconstruct (X,ω) from the coordinates.

More precisely, one can show the following:

Prop.

$\mathcal{TH}_g$ is not connected, unless $g=1$. For any $k=(k_1,\dots,k_n)$ s.t. $\sum_{i=1}^n k_i = 2g-2$, $\mathcal{TH}(k)$ is a union of connected components and it has the structure of a cpx manifold of dim $2g+n-1$.

Note in particular that the dimensions of strata are different. The smallest dim possible is $2g$, when $n=1$, i.e. $\exists!$ zero of ω, and the largest is for $n=2g-2$, when there is the maximum number of zeroes, all of order 1; in this case the dim is $4g-3$. Rmk: these are all cpx dim!

Recall that the dim of $\mathcal{T}_g$ is $3g-3$, which is smaller (for $g\geq 4$) than the dim of $\mathcal{TH}(2g-2)$. This (almost) shows that the maps $\mathcal{TH}(k) \to \mathcal{T}_g$ or $\mathcal{H}(k)\to\mathcal{M}_g$ are not necessarily surjective: they are not projections. Instead, the maps $\mathcal{TH}_g \to \mathcal{T}_g$ and $\mathcal{H}_g \to \mathcal{M}_g$ are surjective: one can show that there is always a nonzero Abelian differential on any Riemann surface X.

One can also describe when strata are connected, but it is much more difficult:

Thm (Kontsevich–Zorich) [Inventiones, 2003]

If $g=2$, all strata ($\mathcal{H}(2)$ and $\mathcal{H}(1,1)$) are connected.

If $g=3$, $\mathcal{H}(4)$ and $\mathcal{H}(2,2)$ have two connected components and all other strata are connected.

If $g\geq 4$:

(1) $\mathcal{H}(2g-2)$ has 3 connected components;

(2) $\mathcal{H}(g-1,g-1)$ has 2 conn. comp. if g even, 3 if g is odd;

(3) $\mathcal{H}(2k_1,\dots,2k_n)$, $n\geq 3$, has 2 connected components;

(4) all other strata are connected.

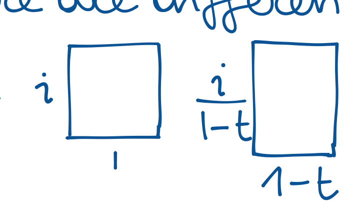
Often one considers translation surfaces normalized to have area 1, where the area is computed with respect to the singular flat metric or as the sum of the areas of a set of polygons defining the translation surface.

We denote by a superscript "1" the parameter spaces restricted to area one surfaces (ex $\mathcal{H}^1_g$ or $\mathcal{TH}^1(k)$).

Prop.

$\mathcal{H}^1(k)$ is never compact.

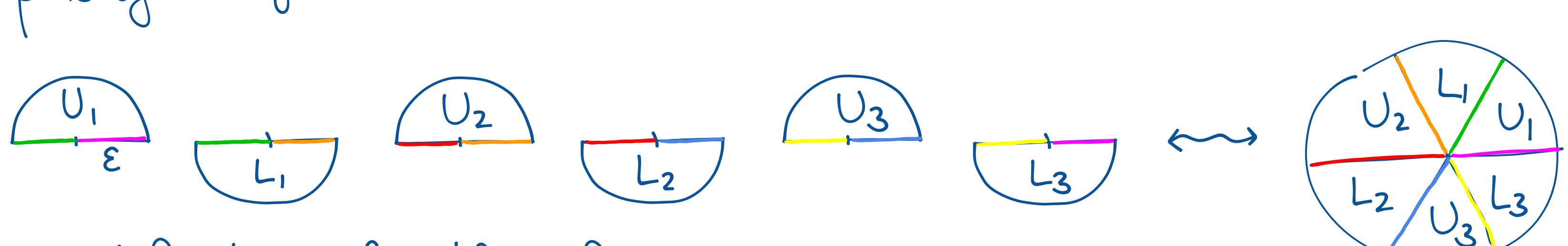
There are different ways in which this non-compactness may arise:

EX.1 

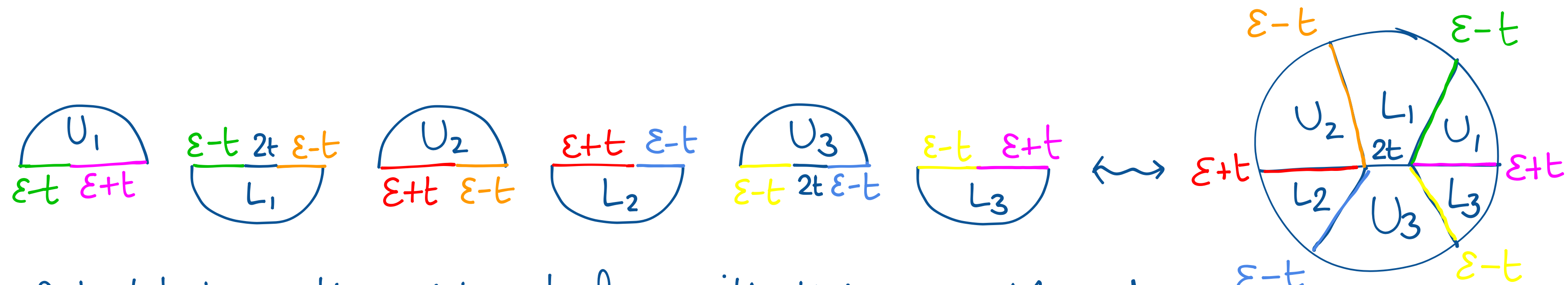
$= X_t \in \mathcal{H}^1(0) = \mathcal{H}^1_1$

As $t\to\infty$, the surface degenerates to a line.

EX.2 Supp. $X\in\mathcal{H}^1(2,k_2,\dots,k_n)$. Let p be the singularity of order 2. A nbhood of p is of the form



We modify this slightly: for $0<t<\varepsilon$



Substituting the nbhood of p with this new nbhood gives us a surface $X_t \in \mathcal{H}^1(1,1,k_2,\dots,k_n)$.

As $t \to 0$, this sequence tends to a surface which is in a different stratum.

Note that in both cases there was a common feature: a curve of length going to zero: a horizontal scc in the torus case and a saddle connection in the other case. This is not a coincidence:

Thm (Masur's compactness criterion)
If $g=1$, a closed subset of $\mathcal{H}_1^1$ is cpt iff there is a positive lower bound on the lengths of all simple closed curves.
If $g \geq 2$, a closed subset of $\mathcal{H}^1(k)$ is cpt iff there is a positive lower bound on the lengths of all saddle connections.