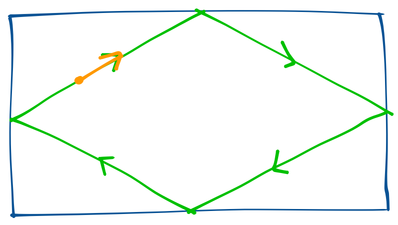
Introduction

Start w/ a billiard table – a rectangle for us. We want to study orbits: pick a point and a direction → what is the trajectory that a ball put at the starting pt hit in the given direction will follow?

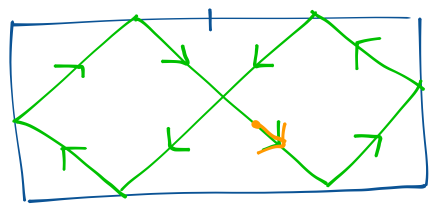
More generally: which kind of trajectories do we get? Periodic, going everywhere, something in between?

Case of a rectangle: easy to draw a periodic orbit

EX 1



EX 2



Rmk: important that we go back to the same point w/ same direction

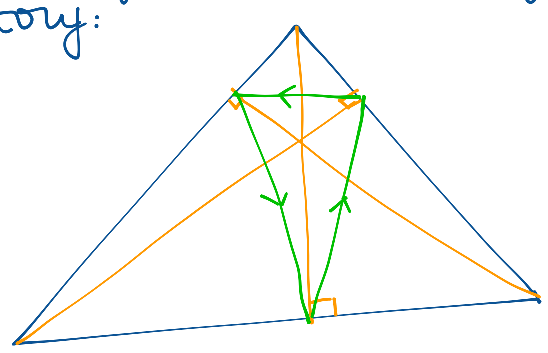
EX 3: start w/ an irrational slope

Claim: this will give a trajectory "going everywhere"

→ we won't prove it now – maybe towards the end

Case of triangles: much less clear how to get a periodic orbit

EX if the triangle has acute angles, there is a periodic orbit, called the Fagnano trajectory:

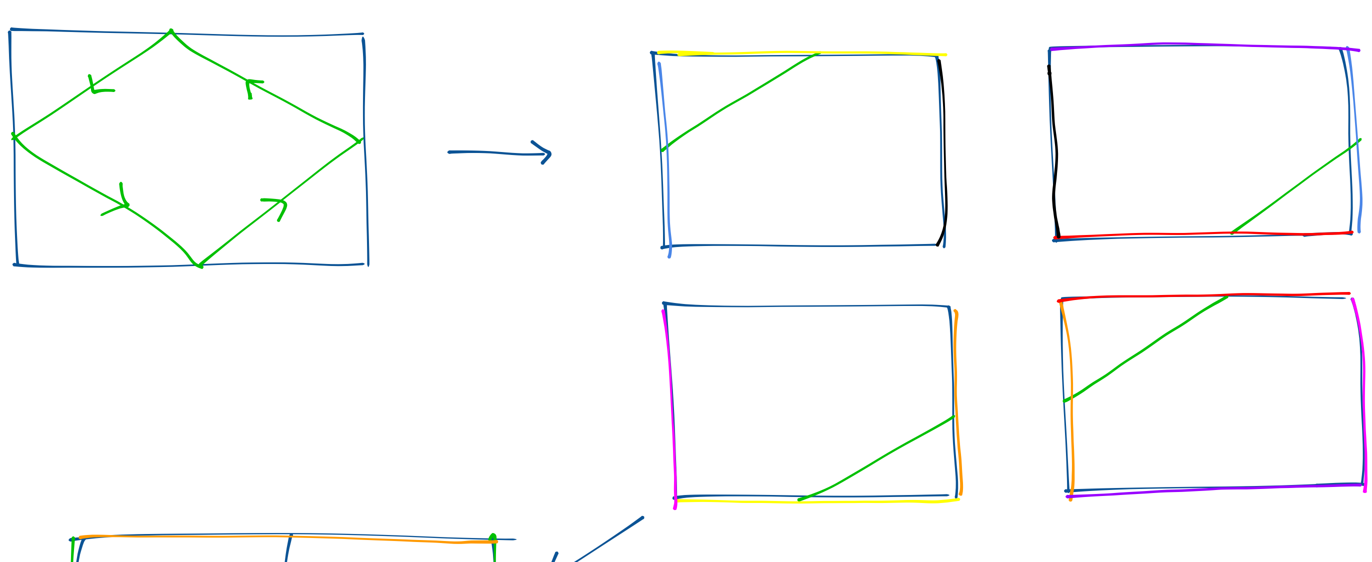


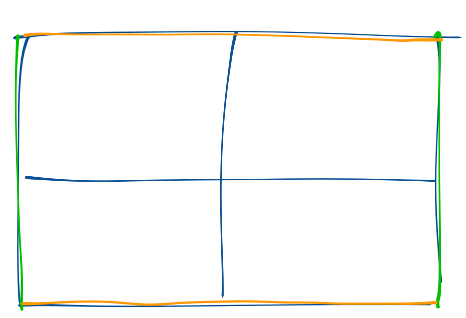
(not obvious, but one can check it)

If the triangle is obtuse, this does not work. Actually in general it is not known if there is a periodic trajectory!

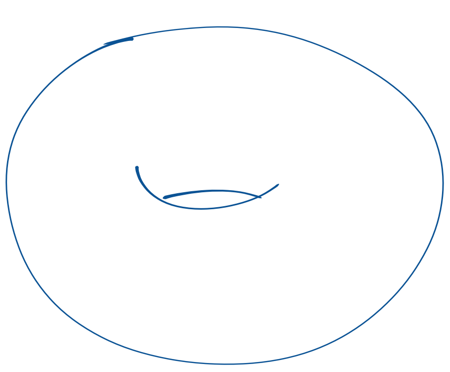
Idea: try to change the viewpt → reflect the billiard instead of the ball

Back to the rectangle case:





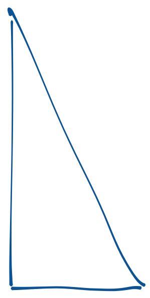
→ torus:



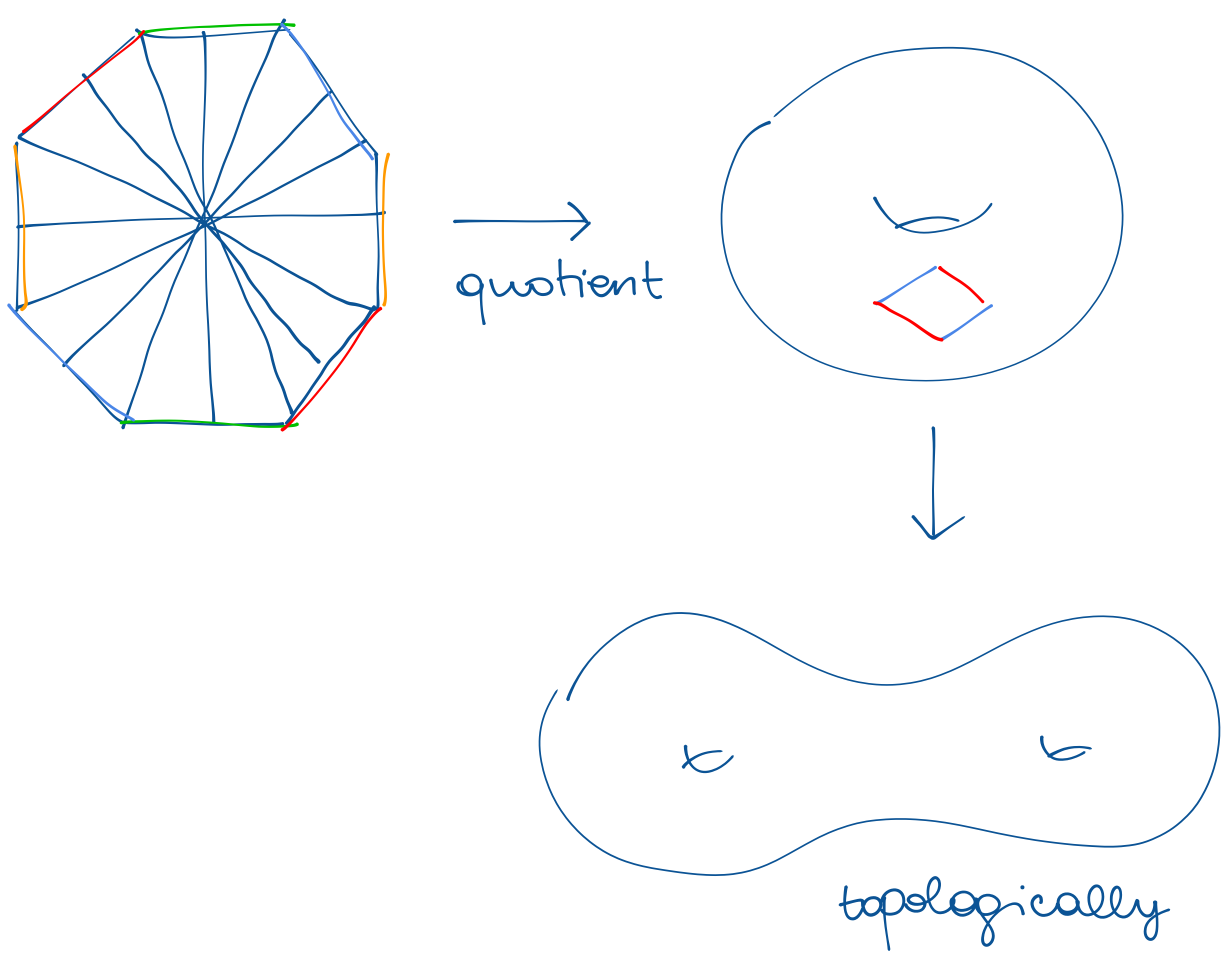
Actually, we get a flat torus, a torus which is locally the same as the Euclidean plane. Periodic billiard orbit ⟷ closed "straight" line on the torus

We can actually do this with any billiard which has angles that are rational multiples of π. It just won't be necessarily as clear.
The main idea is that the number of changes of direction of a trajectory on a polygonal billiard w/ angles that are rational multiples of π is finite, so we can glue together finitely many copies of the polygon, one per direction.

<u>EX</u>: right angled triangle w/ angles $\frac{3\pi}{8}$ and $\pi/8$



Possible directions: given by reflections in sides $\to$ get



Study periodic orbits on <u>these surfaces</u>
$\downarrow$
translation surfaces: surfaces obtained by gluing together polygons using only translations to identify the sides

<u>Thm</u> (Masur)
$\exists$ periodic "orbit" on each such surface.

<u>Cor</u>: $\exists$ closed orbits on billiards w/ angles that are rational multiple of π.