(Infinite) translation surfaces

So far we considered translation surfaces to be compact surfaces with extra structure. What happens if we relax this condition?

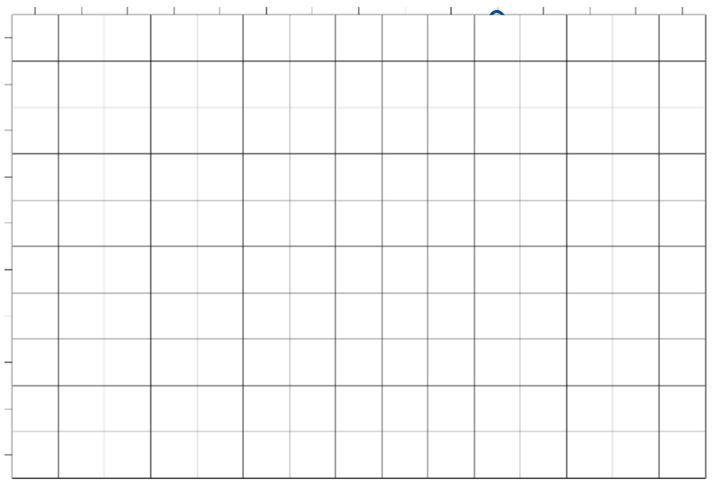
<u>Def</u>. A <u>translation surface</u> X is a surface S with an equivalence class of atlases with translations as transition functions.

<u>Ex</u>: (1) $\mathbb{R}^2$

(2) $X \setminus \Sigma$, where X is one of the translation surfaces considered so far

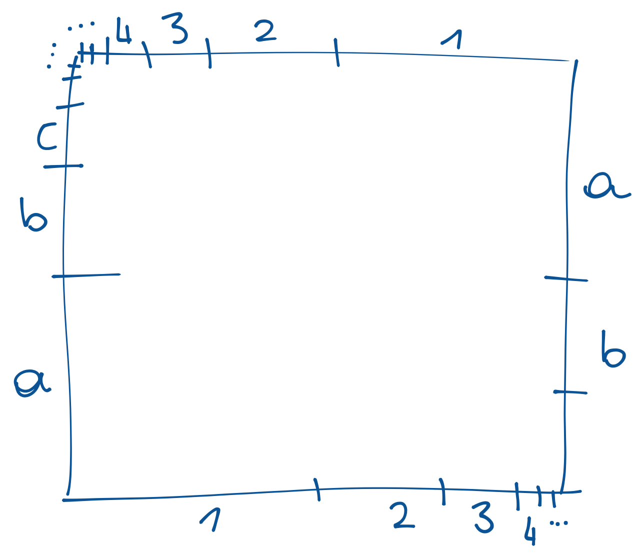
But much more:

(3) <u>infinite staircase</u>



All vertices of the squares are not included

(4) <u>Chamanara surface</u>



All endpts of the segments are excluded

X translation surface $\rightsquigarrow$ it gets a metric $\rightsquigarrow$ we can consider its metric completion $\overline{X} \simeq \{\text{Cauchy sequences in } X\}/\sim$

where $\{x_n\}_n \sim \{y_n\}_n$ if $d_X(x_n, y_n) \longrightarrow 0$ as $n \to \infty$

X is <u>finite</u> if $\overline{X}$ is cpt and $\overline{X} \setminus X$ discrete.

<u>Rmks</u>:
- X open in $\overline{X}$
- $\overline{X}$ cpt $\rightsquigarrow$ $\overline{X} \setminus X$ discrete iff finite.

<u>Singularity</u> = elt of $\overline{X} \setminus X$

$\downarrow$

Three types:

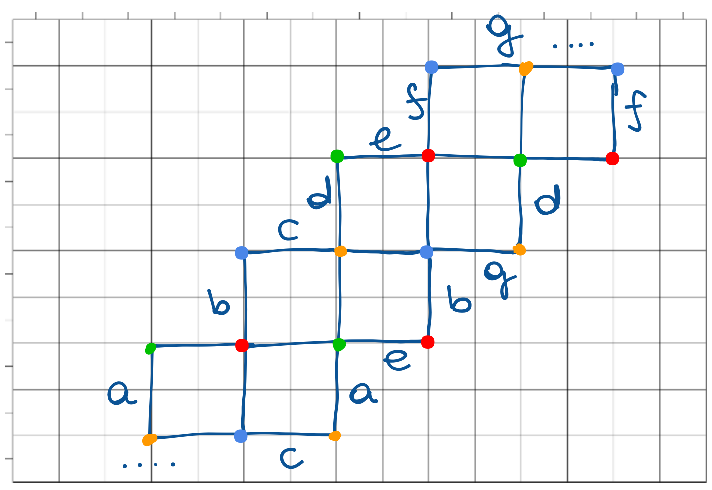
(a) <u>cone points of angle $2\pi k$</u>

$k=1$: removable singularity

(b) pts w/ <u>∞ cone angle</u>: $\exists$ open nbhood B of the singularity p and an $\varepsilon > 0$ s.t. $\exists$ infinite cyclic cover $B \setminus \{p\} \longrightarrow B_\varepsilon(0) \setminus \{0\}$

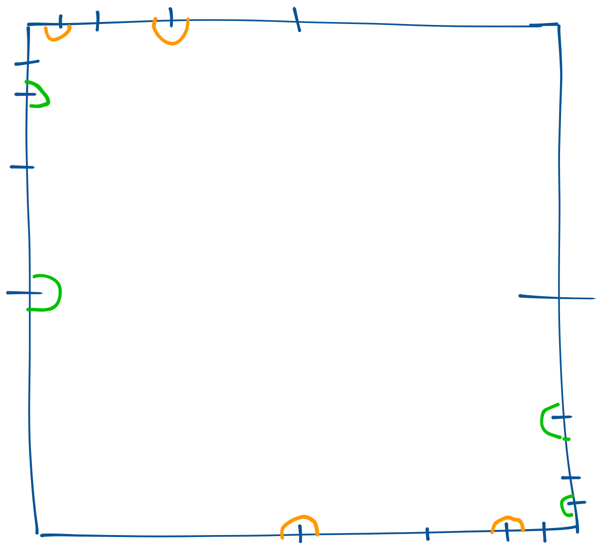
(c) <u>wild</u> singularities: neither of the above

Ex of (b):



4 singularities, all with cone angle ∞

Ex of (c): the unique singularity of the Chamanara surface



We see that the pts surrounded by an arc of the same color are all identified to pts A and B. But the distances of the marked pts go to zero $\Rightarrow$ same in $\overline{X}$ $\rightsquigarrow$ only one singularity

X is <u>tame</u> if it has no wild singularities, <u>wild</u> otherwise.

<u>Rmk</u>: X finite $\Rightarrow$ Gauss-Bonnet tells us the topological type

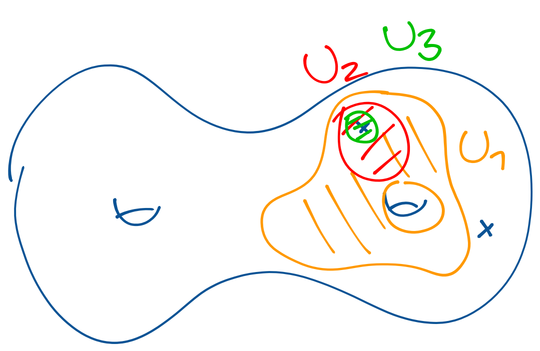
What is the topological type of a possibly infinite translation surface?

$\rightarrow$ general classification of orientable surfaces

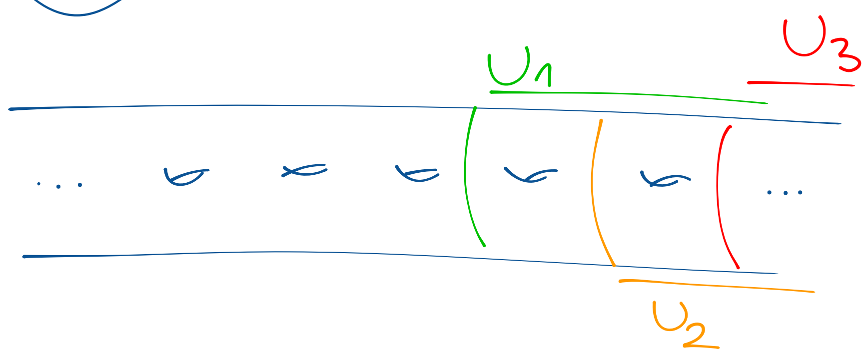
S surface

<u>End</u> of S = [descending chain $U_1 \supseteq U_2 \supseteq \dots$ of open unbounded connected sets with cpt boundary and s.t. $\forall K$ cpt in S, $K \cap U_i = \emptyset$ for all $i \gg 0$] $\longrightarrow$ $U_1 \supseteq U_2 \supseteq \dots \sim V_1 \supseteq V_2 \supseteq \dots$ iff $\forall i \, \exists j : U_i \subseteq V_j$, $\forall k \, \exists l : V_k \subseteq U_l$

<u>Ex.</u>



A puncture (pt removed) is an end.



$\rightarrow$ end accumulated by genus: each U_i has positive genus

$\mathrm{Ends}(S) = \{\text{ends of } S\}$ with topology given by the basis

$$U^* = \{[U_1 \supseteq U_2 \supseteq \dots] \mid U_i \subseteq U \ \forall i \gg 0\}, \quad U \subseteq S \text{ open}$$

$\mathrm{Ends}_g(S) = \{\text{ends accumulated by genus}\}$

<u>Rmk</u>: $\mathrm{Ends}_g(S) \neq \emptyset$ iff genus $= \infty$

Thm (Kerékjártó, Richards)
An or. surface S is top. classified by the triple $(\mathrm{Ends}(S), \mathrm{Ends}_g(S), g)$.
For any E closed subset of the Cantor set, for any F closed subset of E $\exists S$ with $(\mathrm{Ends}(S), \mathrm{Ends}_g(S)) \simeq (E, F)$. Furthermore, the genus of S is ∞ iff $F \neq \phi$; if $F = \phi$, we can choose S with any genus $0 \leq g < \infty$.

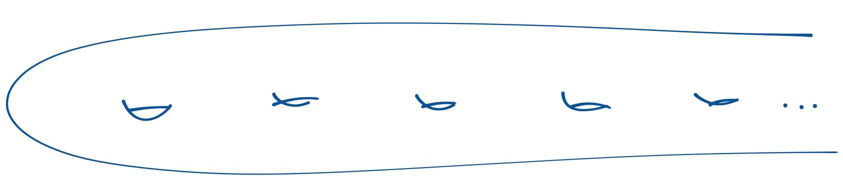
Thm (Randecker)
$F \subseteq E \neq \phi$ closed subsets of the Cantor set $\Rightarrow \exists\, X$ translation surface with only cone angle singularities s.t. $(\mathrm{Ends}(\overline{X}), \mathrm{Ends}_g(\overline{X})) \simeq (E, F)$.

finite ones!

EX: unfolding an irrational billiard

angles not in $\pi\mathbb{Q}$

Thm (Valdez)
The translation surface obtained by unfolding an irrational simply connected polygon is homeomorphic to the Loch Ness monster:



Only one end, accumulated by genus.

Veech group: $\{\text{linear parts of } \mathrm{Aff}_+(X)\} < GL_+(2,\mathbb{R})$
$\mathrm{Aff}_+(X) = \{\text{affine orientation preserving homeomorphisms}\}$

Case X finite: Veech group $= \mathrm{Stab}(X) < SL(2,\mathbb{R})$
Infinite case: not necessarily in $SL(2,\mathbb{R})$!
Set $\mathcal{U} = \{A \in GL_+(2,\mathbb{R}) \mid \|Av\| \leq \|v\| \ \forall v \in \mathbb{R}^2\}$, $P = \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} \middle| \ t \in \mathbb{R}, s > 0 \right\}$ and $P' = \langle P, -\mathrm{Id} \rangle$.

Thm (Przytycki – Schmithüsen – Valdez)
The Veech group G of a tame translation surface satisfies one of the following:
(i) G is countable and disjoint from $\mathcal{U}$;
(ii) G conjugate to P;
(iii) G is conjugate to P';
(iv) $G = GL_+(2,\mathbb{R})$.
Furthermore $\forall\, G < GL_+(2,\mathbb{R})$ satisfying (i), (ii) or (iii) $\exists$ tame transl. surface homeomorphic to the Loch Ness monster whose Veech group is G.

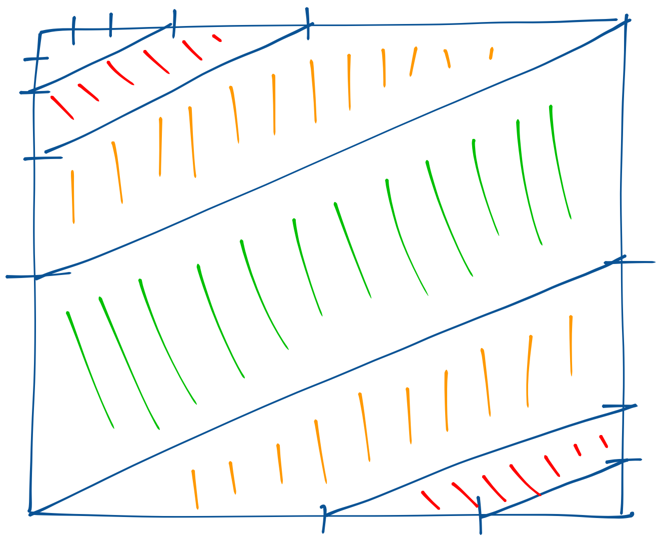
Rmk: any discrete subgroup $G < SL(2,\mathbb{R})$ with area$(\mathbb{H}^2/G) < \infty$ can be realized – it can be cocpt.

An example: the Chamanara surface

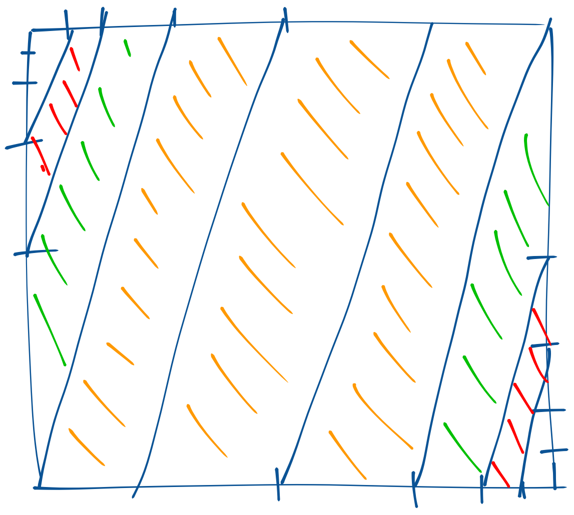
$\downarrow$

cylinder decomposition in direction with slope 2^n for any $n \in \mathbb{Z}$

Ex: slope $\frac{1}{2}$



Ex: slope 4



For each decomposition one can show that the moduli of cylinders given by two trapezoids are the same and the modulus of the middle cylinder is a rational multiple of the others $\Rightarrow$ get many elts in the Veech group.

One can show:

Prop. (Chamanara, Herrlich–Randecker)
The Veech group of $R_{-\pi/4} \cdot$ Chamanara surface is a discrete subgroup of $PSL(2,\mathbb{R})$ generated by $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ and $\frac{1}{4}\begin{pmatrix} -5 & 27 \\ -3 & 13 \end{pmatrix}$.